THE ORDER OF A 2-SEQUENCE AND THE COMPLEXITY OF DIGITAL IMAGES

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The concept of the order of a 2-sequence is introduced in this paper. The order of a 2-sequence is a natural but not trivial extension of the order of one-dimensional (1D) linear recurrent sequences. Necessary and sufficient conditions for the generation of 2-sequences with finite order from the minimal information subset are derived. It is demonstrated that the order of 2-sequences can be used to estimate the complexity of self-organizing patterns with respect to each spatial coordinate.

Keywords: Linear recurrent sequence; Hankel matrix; algebraic decomposition; image complexity.

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1. Introduction

Recurrent sequences play a central role in a large variety of mathematical algorithms and applications. Some of the best known examples of recurrent sequences are used in computational biology. The logistic map was used to model a population growth [20]. The logistic map is often used to illustrate how complex behavior can arise from very simple equations [38], to model [5, 8], to predict [22, 24], to encrypt [27, 37] different physical systems and processes.

Optimal estimation of recurrence structures in neurophysiological time series obtained from anesthetized animals is used to classify the subject's state of consciousness in [1]. Recurrences are widely applied in the theory of recurrence plots, which is a powerful technique for the visualization of the behavior of dynamical systems in phase space [6, 19]. It is shown in [39] that a population code integrating distance naturally emerges in the hippocampus in the form of recurring sequences.

Models incorporating linear recurrent sequences (LRS) are widely used in digital signal processing for system identification when given a sequence of output data and a realization of an underlying state-space model is desired. A first solution to this challenging system-theoretic problem that became as the state-space realization problem was provided in 1965 in [12]. The key tool for solving this problem is the Hankel matrix, whose factorization into the product of an observability matrix and controllability matrix is known as the Ho–Kalman realization method [12]. It took years of research to go from the theoretical results described in [12] to a numerically reliable realization algorithm [4]. Gathering outputs from an impulse-response simulation into a generalized Hankel matrix and its singular value decomposition (SVD) helped to obtain reduced order models for high-dimensional linear dynamical systems [11].

LRS are widely used in analysis of algorithms [3]. The running time of an algorithm can be described in a recurrence relation if it can be broken into smaller subroutines [36]. LRS are also widely used in economics where the functionality of financial sectors depends on lagged variables [34]. LRS are successfully exploited for time series analysis [32] and the construction of solutions to nonlinear ordinary differential equations [26].

The classical one-dimensional (1D) LRS $(x_0, x_1, x_2, \ldots)$ is defined by the linear relation [7]:

$$x_{j+n} = \alpha_1 x_{j+n-1} + \alpha_2 x_{j+n-2} + \cdots + \alpha_n x_j, \tag{1}$$

where $\alpha_k \in \mathbb{R}$, $k = 1, \ldots, n$. Given the initial values $x_0, \ldots, x_{n-1}$, each subsequent term is determined according to (1).

There is a number of generalizations of 1D recurrent sequences to two or more dimensions. Prunescu considers recurrent two-dimensional (2D) sequences over the finite field $\mathcal{A}$ in [30]: given vectors $\lambda \in \mathcal{A}^n$, $\mu \in \mathcal{A}^m$; $n, m \in \mathbb{N}$, a recurrent 2D sequence is defined as the mapping $a : \mathbb{N} \times \mathbb{N} \to \mathcal{A}$, where

(1) $\forall\, i \geq 0 : a(i, 0) = \lambda_{i \bmod n}$;
(2) $\forall\, j \geq 0 : a(0, j) = \mu_{j \bmod m}$;
(3) $\forall\, i, j \geq 1 : a(i, j) = f(a(i-1, j), a(i-1, j-1), a(i, j-1))$, where $f : \mathcal{A}^3 \to \mathcal{A}$.

It is demonstrated that such 2D recurrent sequences can be produced by context-free substitutions and can generate realizations of well-known fractals in [28]. It is shown that in the case $\mathcal{A} = \mathbb{K}$, where $\mathbb{K}$ is Klein's four-element group (the smallest noncyclic group) and f is a linear function

$$f(x, y, z) = Ax + By + Cz, \quad A, B, C = \text{const.}, \tag{2}$$

the resulting recurrent 2D sequences can be classified into 90 groups by their geometric content [29]. A summary of these results can be found in [31].

Multidimensional LRS and linear recurrent arrays over quasi-Frobenius rings and modules are discussed in [18] and [21], respectively. It is demonstrated that n-dimensional LRS over a module can be expressed in the canonical form using eigenvalues of Hankel matrices that are constructed from the sequence [14].

The main objective of this paper is to present an alternative definition of 2D LRS (we will refer to them as 2-LRS) over the field of complex numbers $\mathbb{C}$, utilizing only 1D LRS. It is shown that the existence of the canonical expression of a 2-sequence is necessary and sufficient for the 2-sequence to be linearly recurrent. Furthermore, it is shown that the entire 2-LRS can be reconstructed from the minimal information subset.

The extension of 1D LRS to two dimensions could open new possibilities for the analysis of digital images. One of the objectives of this paper is to develop a new technique for the evaluation of image complexity based on the generalization of the canonical expression theorem for 2-sequences.

The paper is organized as follows: preliminary definitions and properties are discussed in Sec. 2, the proof of the canonical expression theorem and the derivation of the minimal information subset are given in Sec. 3, the application of the concept of 2-LRS for the analysis of self-organizing patterns is presented in Sec. 4, concluding remarks are given in the final section.

2. Preliminaries

2.1. *Definition and properties of LRS*

2.1.1. *General LRS*

Let R be a commutative ring. Any function $P : \mathbb{Z}_0 \to R$ is called a sequence over the ring R and the set of all sequences is denoted $R^{\langle 1 \rangle}$. The elements of the sequence are denoted as $p_j, j \in \mathbb{Z}_0$ and the sequence itself is denoted as $P = (p_j, j \in \mathbb{Z}_0)$. The product of a polynomial $f(\lambda) = \sum_{s=0}^{K} f_s \lambda^s \in R[\lambda]$ and a sequence $P \in R^{\langle 1 \rangle}$ is defined as:

$$f(\lambda)P = \nu, \quad \nu \in R^{\langle 1 \rangle}, \quad \nu_k = \sum_{s \geq 0} f_s p_{s+k}. \tag{3}$$

Definition 1. A sequence $P \in R^{\langle 1 \rangle}$ is called an order m LRS (1-LRS) over R if there exists a monic polynomial $f(\lambda) \in R[\lambda]$ of order m such that $f(\lambda)P = 0$. The polynomial $f(\lambda)$ is called the characteristic polynomial of P and the first m values of the sequence $(p_0, p_1, \ldots, p_{m-1})$ are called the initial vector of P [14].

For example, the Fibonacci sequence is second-order 1-LRS over $\mathbb{Z}$. It's characteristic polynomial and initial vector are $x^2 - x - 1$ and $(0, 1)$, respectively.

Note that the elements of 1-LRS can be computed using n preceding elements of the sequence [7]:

$$p_{k+n} = s_{n-1} p_{k+n-1} + \cdots + s_0 p_k. \tag{4}$$

A function $X : \mathbb{Z}_0^2 \to R$ is called a 2-sequence and the set of all 2-sequences over R is denoted $R^{\langle 2 \rangle}$. In the context of this paper, a 2-sequence can be considered as infinite matrix or data array, with elements denoted $x_{kl}; k, l \in \mathbb{Z}_0$ and the 2-sequence itself denoted $X = [x_{jr}]_{j,r=0}^{+\infty}$.

Consider a bivariate polynomial $f(\lambda, \mu) = \sum_{s=0}^{K} \sum_{t=0}^{L} f_{s,t} \lambda^s \mu^t \in R[\lambda, \mu]$. The product of a polynomial and a 2-sequence is defined as:

$$f(\lambda, \mu)X = \nu, \quad \nu \in R^{\langle 2 \rangle}, \quad \nu_{k,l} = \sum_{s,t \geq 0} f_{s,t} x_{k+s,l+t}. \tag{5}$$

Definition 2. A 2-sequence $X \in R^{\langle 2 \rangle}$ is called a 2-LRS if there exist monic polynomials $f_1(\lambda), f_2(\mu)$ such that $(f_1(\lambda) f_2(\mu))X = 0$ [14].

Theorem 1 (Canonical form of 1-LRS and 2-LRS). *Suppose the characteristic polynomial of* 1-*LRS can be written as*:

$$f(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_l)^{n_l}, \tag{6}$$

then the elements of that sequence can be expressed in the canonical form:

$$p_j = \sum_{k=1}^{l} \sum_{s=0}^{m_k - 1} \xi_{ks} \binom{j}{s} \lambda_k^{j-s}, \tag{7}$$

where $\lambda_k, \xi_{ks} \in R$.

Suppose the characteristic polynomial of a 2-*LRS can be written as*:

$$f_1(\lambda) f_2(\mu) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_n)^{n_n} (\mu - \mu_1)^{m_1} \cdots (\mu - \mu_m)^{m_m}, \tag{8}$$

then the elements of that 2-*sequence can be expressed in the canonical form*:

$$x_{jr} = \sum_{k=1}^{n} \sum_{s=0}^{n_k - 1} \sum_{l=1}^{m} \sum_{t=0}^{m_l - 1} c_{kl}^{(st)} \binom{r}{s} \binom{j}{t} \lambda_k^{r-s} \mu_l^{j-t}, \tag{9}$$

where $\lambda_k, \mu_l, c_{kl}^{(st)} \in R$.

The coefficients present in both canonical forms, ξ_{ks} and $c_{kl}^{(st)}$ respectively, are determined to fit the initial conditions of the recurrences.

2.1.2. 1-*LRS over* $\mathbb{C}$

In this and all subsequent sections of this paper, we will deal with linear recurrent sequences over the complex field, thus $R = \mathbb{C}$. For 1-LRS over $\mathbb{C}$ there is a convenient criterion based on the Hankel matrix which simplifies the determination of the order of the sequence.

Let us consider a complex-valued sequence P. Using P, a sequence of Hankel matrices $(H_n; n \in \mathbb{N})$ can be formed:

$$H_n := \begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} \\ p_1 & p_2 & \cdots & p_n \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_n & \cdots & p_{2n-2} \end{bmatrix}. \tag{10}$$

The Hankel mapping $(d_n; n \in \mathbb{N})$ reads:

$$d_n := \det(H_n).$$

Definition 3. The sequence $P = (p_j; j \in \mathbb{Z}_0)$ is an order $m \in \mathbb{Z}_0; (m < +\infty)$ 1-LRS over $\mathbb{C}$, if the Hankel mapping of that sequence has the following structure:

$$(d_1, \ldots, d_m, 0, 0, \ldots), \tag{11}$$

where $d_m \neq 0$ and $d_{m+k} = 0$, $k = 1, 2, \ldots$.

Note that Definition 1 with $R = \mathbb{C}$ and Definition 3 are equivalent.

The characteristic polynomial for the order m 1-LRS $(p_j; j \in \mathbb{Z}_0)$ can be expressed as:

$$f(\lambda) = \begin{bmatrix} p_0 & p_1 & \cdots & p_m \\ p_1 & p_2 & \cdots & p_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m-1} & p_m & \cdots & p_{2m-1} \\ 1 & \lambda & \cdots & \lambda^m \end{bmatrix}. \tag{12}$$

Expanding the determinant in (12) yields an mth order polynomial:

$$f(\lambda) = A_m \lambda^m + \cdots + A_1 \lambda + A_0, \tag{13}$$

with $A_m \neq 0$ according to (11).

Theorem 2 (Canonical form of 1-LRS over $\mathbb{C}$). *Suppose $P = (p_j; j \in \mathbb{Z}_0)$ is an order m 1-LRS and the roots of its characteristic polynomial are $\lambda_1, \ldots, \lambda_l$ with multiplicities $m_1, \ldots, m_l$. Then $\sum_{k=1}^{l} m_k = m$ and the elements of P can be expressed in the canonical form:*

$$p_j = \sum_{k=1}^{l} \sum_{s=0}^{m_k - 1} \xi_{ks} \binom{j}{s} \lambda_k^{j-s}, \tag{14}$$

where λ_k, $\xi_{ks} \in \mathbb{C}$; $\xi_{k,m_k-1} \neq 0$. Note that $0^0 := 1$ and $\binom{j}{s}\lambda^k = 0$ if at least one of the factors is zero.

The reversed statement is also true. If (14) holds, then $P = (p_j; j \in \mathbb{Z}_0)$ is a 1-LRS of order m.

Note that in the general case, sequences defined by the expression (7) are not necessarily 1-LRS, but over the complex field, sequences are 1-LRS if and only if (14) holds [25].

Remark 1. The coefficients ξ_{ks} in (14) are determined in order to fit the initial conditions of the recurrence:

$$\sum_{k=1}^{l}\sum_{s=0}^{m_k-1}\xi_{ks}\binom{j}{s}\lambda_k^{j-s}=p_j; \quad j=0,\dots,m-1, \tag{15}$$

where the characteristic roots $\lambda_1,\dots,\lambda_l$ are known. This system has one and only one solution [25].

Remark 2. Suppose that $P=(p_j; j\in\mathbb{Z}_0)$ is an order m 1-LRS and the first $2m$ elements are known. Then, using (13), (14) and (15) all elements of that sequence can be determined.

Remark 3. Suppose that $P=(p_j; j\in\mathbb{Z}_0)$ is an order m 1-LRS and $Q=(p_{j+k}; j\in\mathbb{Z}_0), k\in\mathbb{Z}_0$. Then, Q is an order m_q 1-LRS, where $m_q\le m$.

2.2. 2-*sequences*

This section is dedicated to the definition of 2-sequences and introduction of some notational conventions. Complex 2-sequences $X:=[x_{jr}]_{j,r=0}^{+\infty}$, where $x_{jr}\in\mathbb{C}$ are considered.

Any 2-sequence has two elementary families of 1-sequences:

$$R_k(X):=(x_{kr}; r\in\mathbb{Z}_0), \tag{16}$$

for fixed $k\in\mathbb{Z}_0$ is called the kth *row sequence* of X. Likewise,

$$C_l(X):=(x_{jl}; j\in\mathbb{Z}_0), \tag{17}$$

for fixed $l\in\mathbb{Z}_0$ is called the lth *column sequence* of X.

Example 1. Let $X=[x_{jr}]_{j,r=0}^{+\infty}$, where $x_{j0}=(j+2)!, x_{jr}=1+x_{j,r-1}; j=0,1,\dots;$ $r=1,2,\dots$. It can be seen that:

$$X=\begin{bmatrix} 2 & 3 & 4 & 5 & \cdots \\ 6 & 7 & 8 & 9 & \cdots \\ 24 & 25 & 26 & 27 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The row sequences $R_j(X)=((j+2)!, (j+2)!+1,\dots,(j+2)!+r,\dots)$ are order 2 1-LRS for $j=0,1,\dots$. Since all of the row sequences have a single characteristic root $\lambda_1 = 1$ and its multiplicity is $n_1 = 2$, elements of X can be expressed as:

$$x_{jr} = \xi_{10}^{(j)} + \xi_{11}^{(j)} \binom{r}{1}.$$

Coefficients $\xi_{10}^{(j)}$, $\xi_{11}^{(j)}$ can be found using the first two elements of the jth row of X. For example

$$\xi_{10}^{(0)} + \xi_{11}^{(0)} \binom{0}{1} = 2;$$

$$\xi_{10}^{(0)} + \xi_{11}^{(0)} \binom{1}{1} = 3,$$

yields $\xi_{10}^{(0)} = 2$, $\xi_{11}^{(0)} = 1$. It can be verified that $\xi_{10}^{(j)} = (j+2)!$, $\xi_{11}^{(j)} = 1$, thus the coefficients differ for each $j = 0, 1, \ldots$.

3. The Order of a 2-Sequence

In this section, the order of a 2-sequence based on the concept of 1-LRS is introduced. The main goal of this section is to present the canonical form and the minimal information subset of linear recurrent 2-sequences.

3.1. *Row and column order of 2-sequences*

Definition 4. Let $X = [x_{jr}]_{j,r=0}^{+\infty}$. Suppose that each row sequence $R_k(X)$, $k = 0, 1, \ldots$ is a 1-LRS and Λ is the finite set of characteristic roots in row sequences $R_k(X)$, $k = 0, 1, \ldots$ (omitting repetitions of roots). Thus $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $n < +\infty$. The multiplicity of $\lambda_k \in \Lambda$ is defined as $n_k = \max_{j \geq 0} n_k^{(j)} < +\infty$, where $n_k^{(j)}$ is the multiplicity of λ_k in sequence $R_j(X)$ (if λ_k is not a root of the characteristic polynomial corresponding to $R_j(X)$, then $n_k^{(j)} := 0$). If these conditions are met, X has a row order equal to $N = \sum_{k=1}^{n} n_k$.

Elements of set Λ are called the row characteristic roots of X.

Remark 4. Suppose X has a row order of N and its row characteristic roots are $\lambda_1, \ldots, \lambda_n$ with multiplicities $n_1, \ldots, n_n$. Then, by Theorem 2, elements of X can be written as:

$$x_{jr} = \sum_{k=1}^{n} \sum_{s=0}^{n_k - 1} \xi_{ks}^{(j)} \binom{r}{s} \lambda_k^{r-s}, \tag{18}$$

where coefficients $\xi_{ks}^{(j)} \in \mathbb{C}$ are different for each j in the general case.

Definition 5. Let $X = [x_{jr}]_{j,r=0}^{+\infty}$. Suppose that each column sequence $C_l(X)$, $l = 0, 1, \ldots$ is a 1-LRS and $\mathcal{M}$ is the finite set of characteristic roots in column sequences $C_l(X)$, $l = 0, 1, \ldots$ (omitting repetitions of roots). Thus, $\mathcal{M} = \{\mu_1, \mu_2, \ldots, \mu_m\}$, $m < +\infty$ and the multiplicity of each $\mu_l \in M$ is defined as $m_l = \max_{r \geq 0} m_l^{(r)} <$

$+\infty$, where $m_l^{(r)}$ is the multiplicity of μ_l in sequence $C_r(X)$ (if μ_l is not a root of the characteristic polynomial corresponding to $C_r(X)$ then $m_l^{(r)} := 0$). If these conditions are met, X has a column order equal to $M = \sum_{l=1}^{m} m_l$.

Elements of set $\mathcal{M}$ are called the column characteristic roots of X.

Remark 5. Similarly to Remark 4, if column characteristic roots of X are $\mu_1, \ldots, \mu_m$ with multiplicities $m_1, \ldots, m_m$, elements of X can be expressed as:

$$x_{jr} = \sum_{l=1}^{m} \sum_{t=0}^{m_l - 1} \eta_{lt}^{(r)} \binom{j}{t} \mu_l^{j-t}, \tag{19}$$

where coefficients $\eta_{lt}^{(r)} \in \mathbb{C}$ are different for each r in the general case.

Definition 6. A 2-sequence X has a 2D order ord_2 if it has both row order and column order. It is denoted as:

$$\mathrm{ord}_2 X = (N, M), \tag{20}$$

where N is the row order and M is the column order of X. If X only has finite row or column order, it is denoted as:

$$\mathrm{ord}_2 X = (N, +\infty); \quad \mathrm{ord}_2 X = (+\infty, M),$$

respectively. If X does not have finite row and column order, the notation is

$$\mathrm{ord}_2 X = (+\infty, +\infty).$$

Note that it is sufficient to have $\mathrm{ord}_2 X = (N, +\infty)$ for (18) to hold and $\mathrm{ord}_2 X = (+\infty, M)$ for (19) to hold. In Example 1, $\mathrm{ord}_2 X = (2, +\infty)$, because the factorial sequence is not a 1-LRS. An example of a 2-sequence with infinite order is given by $X = [x_{jr}]_{j,r=0}^{+\infty}$, where $x_{jr} = (jr)!$.

Example 2. Suppose $X = [x_{jr}]_{j,r=0}^{+\infty}$, where $x_{jr} = (j+1)(r+2)^2$. Then, the row order of X is 3 and the column order is 2, thus $\mathrm{ord}_2 X = (3, 2)$.

Note that it is not sufficient for the characteristic root sets $\Lambda, \mathcal{M}$ of to be finite for a 2-sequence to satisfy Definition 6. Special attention should be directed to how the multiplicities of the characteristic roots are calculated.

Example 3. Let $D = [\delta_{jr}]_{j,r=0}^{+\infty}$, where δ_{jr} is the Kronecker delta.

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{21}$$

It can be observed that $R_k(D) = C_k(D) = (0, 0, \ldots, 0, 1, 0, \ldots)$ where only the kth element is nonzero. The order of $R_k(D)$ and $C_k(D)$ is k. However, since the row sequences $R_k(D)$ have a single characteristic root $\lambda_1 = 1$ with multiplicity

$n_1^{(k)} = k$, the maximum multiplicity in all row sequences of D of characteristic root λ_1 is $n_1 = \max_{k\geq 0} n_1^{(k)} = \max_{k\geq 0} k = +\infty$, thus the conditions of Definition 4 are not met and D has an infinite row order. Since in this case the kth column and row sequences are congruent, D also has an infinite column order and thus $\mathrm{ord}_2 D = (+\infty, +\infty)$.

Remark 6. Suppose that $\mathrm{ord}_2 X = (N, M)$, where $X = [x_{jr}]_{j,r=0}^{+\infty}$. Let $Y = [y_{jr}]_{j,r=0}^{+\infty}$, where $y_{jr} := x_{j+k,r+l}$; $k, l \in \mathbb{Z}_0$, then $\mathrm{ord}_2 Y = (N_1, M_1)$, with $N_1 \leq N$ and $M_1 \leq M$.

Proof. By Remark 3, the order of $R_k(Y)$ cannot exceed the order of $R_k(X)$ and the order of $C_l(Y)$ cannot exceed the order $C_l(X)$. Since the orders of both rows and columns of Y do not exceed the corresponding orders of X, it can be concluded that $\mathrm{ord}_2 Y = (N_1, M_1)$ and $N_1 \leq N$, $M_1 \leq M$. □

3.2. *Canonical form theorem*

In this section, it is shown that a 2-sequence has finite order if and only if it can be written in the canonical form.

Theorem 3. *Suppose that* $X = [x_{jr}]_{j,r=0}^{+\infty}$ *is a 2-sequence with* $\mathrm{ord}_2 X = (N, M)$. *Then any element of* X *can be expressed as*:

$$x_{jr} = \sum_{k=1}^{n} \sum_{s=0}^{n_k-1} \sum_{l=1}^{m} \sum_{t=0}^{m_l-1} c_{kl}^{(st)} \binom{r}{s} \binom{j}{t} \lambda_k^{r-s} \mu_l^{j-t}, \tag{22}$$

where λ_k, $k = 1, \ldots, n$; μ_l, $l = 1, \ldots, m$ *are the row and column characteristic roots respectively, with multiplicities* n_k, $k = 1, \ldots, n$ *and* $m_l, l = 1, \ldots, m$; $c_{kl}^{(st)} \in \mathbb{C}$ *are constants which do not depend on* j *and* r.

Proof. Remarks 4 and 5 state that if $\mathrm{ord}_2 X = (N, M)$ then any element x_{jr} of X can be expressed as:

$$x_{jr} = \sum_{k=1}^{n} \sum_{s=0}^{n_k-1} \xi_{ks}^{(j)} \binom{r}{s} \lambda_k^{r-s}; \tag{23}$$

or

$$x_{jr} = \sum_{l=1}^{m} \sum_{t=0}^{m_l-1} \eta_{lt}^{(r)} \binom{j}{t} \mu_l^{j-t}, \tag{24}$$

where $\xi_{ks}^{(j)}$ and $\eta_{lt}^{(r)}$ are constants dependent on j and r, respectively. Equation (22) can be rewritten as:

$$x_{jr} = \sum_{k=1}^{n} \sum_{s=0}^{n_k-1} \binom{r}{s} \lambda_k^{r-s} \sum_{l=1}^{m} \sum_{t=0}^{m_l-1} c_{kl}^{(st)} \binom{j}{t} \mu_l^{j-t}; \tag{25}$$

or

$$x_{jr} = \sum_{l=1}^{m} \sum_{t=0}^{m_l-1} \binom{j}{t} \mu_l^{j-t} \sum_{k=1}^{n} \sum_{s=0}^{n_k-1} c_{kl}^{(st)} \binom{r}{s} \lambda_k^{r-s}. \tag{26}$$

Equating (23) and (25) systems of linear equations for the determination of $c_{kl}^{(st)}$ are obtained:

$$\sum_{l=1}^{m} \sum_{t=0}^{m_l-1} \binom{j}{t} \mu_l^{j-t} c_{kl}^{(st)} = \xi_{ks}^{(j)}, \tag{27}$$

where $k = k_0 \in \{1, \ldots, n\}$, $s = s_0 \in \{0, 1, \ldots, n_{k_0} - 1\}$ are fixed and $j = 0, 1, \ldots, M-1$. The number of linear systems required to determine all coefficients $c_{kl}^{(st)}$ is equal to N.

Note that the matrices of all the linear systems defined in (27) are equal to the confluent Vandermonde matrix [10]. Thus (27) has a single solution, denoted as $\hat{c}_{kl}^{(st)}$.

The coefficients $c_{kl}^{(st)}$ can also be calculated by solving M linear systems of order N, that are obtained analogously to (27). Equating (24) and (26) yields:

$$\sum_{k=1}^{n} \sum_{s=0}^{n_k-1} \binom{r}{s} \lambda_k^{r-s} c_{kl}^{(st)} = \eta_{lt}^{(r)}, \tag{28}$$

where $l = l_0 \in \{1, \ldots, m\}$, $t = t_0 \in \{0, 1, \ldots, m_{l_0} - 1\}$ are fixed and $r = 0, 1, \ldots, N-1$.

The matrices of these linear systems are congruent for all l and t and are equal to the confluent Vandermonde matrix. The solution to (28) is denoted as $\tilde{c}_{kl}^{(st)}$.

Because $\hat{c}_{kl}^{(st)}$ and $\tilde{c}_{kl}^{(st)}$ satisfy (27) and (28) respectively, (25) and (26) yields:

$$\begin{aligned} x_{jr} &= \sum_{k=1}^{n} \sum_{s=0}^{n_k-1} \sum_{l=1}^{m} \sum_{t=0}^{m_l-1} \hat{c}_{kl}^{(st)} \binom{r}{s} \binom{j}{t} \lambda_k^{r-s} \mu_l^{j-t} \\ &= \sum_{k=1}^{n} \sum_{s=0}^{n_k-1} \sum_{l=1}^{m} \sum_{t=0}^{m_l-1} \tilde{c}_{kl}^{(st)} \binom{r}{s} \binom{j}{t} \lambda_k^{r-s} \mu_l^{j-t}. \end{aligned}$$

Thus $\hat{c}_{kl}^{(st)} = \tilde{c}_{kl}^{(st)}$ for all k, l, s, t, so the coefficients $c_{kl}^{(st)}$ are unique. □

Corollary 1. *The reversed statement to Theorem* 3 *is also true. Suppose that any element of a* 2*-sequence X can be written as in* (22)*. Then* $\mathrm{ord}_2 X = (N, M)$*.*

Proof. By (25), (23) and (27), it can be deduced that X has a row order of N. Analogously, by (26), (24) and (28) X has a column order of M. □

Example 4. Let $X = [x_{jr}]_{j,r=0}^{+\infty}$, where $x_{0r} = \cos(\frac{\pi}{2}r)$, $r = 0, 1, 2, \ldots$ and $x_{jr} = x_{0r} + (j+1)^2$, $j = 1, 2, \ldots$, then:

$$X = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & \cdots \\ 5 & 4 & 3 & 4 & 5 & 4 & \cdots \\ 10 & 9 & 8 & 9 & 10 & 9 & \cdots \\ 17 & 16 & 15 & 16 & 17 & 16 & \cdots \\ 26 & 25 & 24 & 25 & 26 & 25 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{29}$$

By calculating Hankel mappings for the rows of X it can be noted that the row characteristic roots of X are $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_3 = 1$, with multiplicities $n_1 = n_2 = n_3 = 1$. Likewise inspecting the columns of X, it can be seen that the column characteristic roots are $\mu_1 = 0$, $\mu_2 = 1$ with multiplicities $m_1 = 1$, $m_2 = 3$. This means that $\text{ord}_2 X = (3, 4)$.

The coefficients $c_{kl}^{(st)}$ are calculated using (28). To do so, the coefficients $\eta_{lt}^{(r)}$, $r = 0, 1, 2$ must be obtained. They are solutions of the following linear systems of equations:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ \mu_1 & \mu_2 & 1 & 0 \\ \mu_1^2 & \mu_2^2 & 2\mu_2 & 1 \\ \mu_1^3 & \mu_2^3 & 3\mu_2^2 & 3\mu_2 \end{bmatrix} \begin{bmatrix} \eta_{10}^{(r)} \\ \eta_{20}^{(r)} \\ \eta_{21}^{(r)} \\ \eta_{22}^{(r)} \end{bmatrix} = \begin{bmatrix} x_{0r} \\ x_{1r} \\ x_{2r} \\ x_{3r} \end{bmatrix}, \quad r = 0, 1, 2. \tag{30}$$

Since the matrices of the linear systems (30) are equal, they can be written simultaneously:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ \mu_1 & \mu_2 & 1 & 0 \\ \mu_1^2 & \mu_2^2 & 2\mu_2 & 1 \\ \mu_1^3 & \mu_2^3 & 3\mu_2^2 & 3\mu_2 \end{bmatrix} \begin{bmatrix} \eta_{10}^{(0)} & \eta_{10}^{(1)} & \eta_{10}^{(2)} \\ \eta_{20}^{(0)} & \eta_{20}^{(1)} & \eta_{20}^{(2)} \\ \eta_{21}^{(0)} & \eta_{21}^{(1)} & \eta_{21}^{(2)} \\ \eta_{22}^{(0)} & \eta_{22}^{(1)} & \eta_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & x_{22} \\ x_{30} & x_{31} & x_{32} \end{bmatrix}. \tag{31}$$

The solution reads:

$$\begin{bmatrix} \eta_{10}^{(0)} & \eta_{10}^{(1)} & \eta_{10}^{(2)} \\ \eta_{20}^{(0)} & \eta_{20}^{(1)} & \eta_{20}^{(2)} \\ \eta_{21}^{(0)} & \eta_{21}^{(1)} & \eta_{21}^{(2)} \\ \eta_{22}^{(0)} & \eta_{22}^{(1)} & \eta_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 1 & 0 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix}. \tag{32}$$

Using (28), coefficients $c_{kl}^{(st)}$ are obtained:

$$\begin{aligned}
&\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} c_{11}^{(00)} \\ c_{21}^{(00)} \\ c_{31}^{(00)} \end{bmatrix} = \begin{bmatrix} \eta_{10}^{(0)} \\ \eta_{10}^{(1)} \\ \eta_{10}^{(2)} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ i & -i & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_{11}^{(00)} \\ c_{21}^{(00)} \\ c_{31}^{(00)} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}; \\
&\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} c_{12}^{(00)} \\ c_{22}^{(00)} \\ c_{32}^{(00)} \end{bmatrix} = \begin{bmatrix} \eta_{20}^{(0)} \\ \eta_{20}^{(1)} \\ \eta_{20}^{(2)} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ i & -i & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_{12}^{(00)} \\ c_{22}^{(00)} \\ c_{32}^{(00)} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}; \\
&\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} c_{12}^{(01)} \\ c_{22}^{(01)} \\ c_{32}^{(01)} \end{bmatrix} = \begin{bmatrix} \eta_{21}^{(0)} \\ \eta_{21}^{(1)} \\ \eta_{21}^{(2)} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ i & -i & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_{12}^{(01)} \\ c_{22}^{(01)} \\ c_{32}^{(01)} \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}; \\
&\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} c_{12}^{(02)} \\ c_{22}^{(02)} \\ c_{32}^{(02)} \end{bmatrix} = \begin{bmatrix} \eta_{22}^{(0)} \\ \eta_{22}^{(1)} \\ \eta_{22}^{(2)} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ i & -i & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_{12}^{(02)} \\ c_{22}^{(02)} \\ c_{32}^{(02)} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.
\end{aligned} \tag{33}$$

The coefficients $c_{kl}^{(st)}$ can be conveniently represented by a matrix of the following structure:

$$C = \begin{bmatrix} c_{11}^{(00)} & c_{12}^{(00)} & c_{12}^{(01)} & c_{12}^{(02)} \\ c_{21}^{(00)} & c_{22}^{(00)} & c_{22}^{(01)} & c_{22}^{(02)} \\ c_{31}^{(00)} & c_{32}^{(00)} & c_{32}^{(01)} & c_{32}^{(02)} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -1 & 1 & 3 & 2 \end{bmatrix}. \tag{34}$$

Using the data obtained, the expression for any element of X can be written:

$$x_{jr} = \frac{1}{2}1^j i^r + \frac{1}{2}1^j(-i)^r - 0^j 1^r + 1^j 1^r + 3\binom{j}{1}1^{j-1}1^r + 2\binom{j}{2}1^{j-2}1^r. \tag{35}$$

3.3. *Minimal information subset and generation of 2-sequences with finite order*

In this section, it is proven that a finite subset of elements is sufficient to calculate any element of a 2-sequence with finite order.

Definition 7. Suppose $X = [x_{jr}]_{j,r=0}^{+\infty}$ is a 2-sequence with $\operatorname{ord}_2 X = (N, M)$. The elbow of X, denoted $\mathrm{El}_{2N\times 2M}(X)$ is a finite subset of X, given by

$$\begin{aligned} \mathrm{El}_{2N\times 2M}(X) = \{x_{jr} \mid j = 0, \dots, 2M-1; r = 0, \dots, 2N-1\}/ \\ \{x_{jr} \mid j = M, \dots, 2M-1; r = N, \dots, 2N-1\}. \end{aligned} \tag{36}$$

A graphical representation of $\mathrm{El}_{2N\times 2M}(X)$ is given below. Note that the number of elements in $\mathrm{El}_{2N\times 2M}(X)$ is $2N\cdot 2M - NM = 3NM$.

$$\begin{bmatrix}
\boldsymbol{x_{00}} & \boldsymbol{x_{01}} & \boldsymbol{\cdots} & \boldsymbol{x_{0,N-1}} & \boldsymbol{x_{0,N}} & \boldsymbol{\cdots} & \boldsymbol{x_{0,2N-1}} & x_{0,2N} & \cdots \\
\boldsymbol{x_{10}} & \boldsymbol{x_{11}} & \boldsymbol{\cdots} & \boldsymbol{x_{1,2N-1}} & \boldsymbol{x_{1,N}} & \boldsymbol{\cdots} & \boldsymbol{x_{1,2N-1}} & x_{1,2N} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\boldsymbol{x_{M-1,0}} & \boldsymbol{x_{M-1,1}} & \boldsymbol{\cdots} & \boldsymbol{x_{M-1,N-1}} & \boldsymbol{x_{M-1,N}} & \boldsymbol{\cdots} & \boldsymbol{x_{M-1,2N-1}} & x_{M-1,2N} & \cdots \\
\boldsymbol{x_{M,0}} & \boldsymbol{x_{M,1}} & \boldsymbol{\cdots} & \boldsymbol{x_{M,N-1}} & \boldsymbol{x_{M,N}} & \boldsymbol{\cdots} & \boldsymbol{x_{M,2N-1}} & x_{M,2N} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\boldsymbol{x_{2M-1,0}} & \boldsymbol{x_{2M-1,1}} & \boldsymbol{\cdots} & \boldsymbol{x_{2M-1,N-1}} & \boldsymbol{x_{2M-1,N}} & \boldsymbol{\cdots} & \boldsymbol{x_{2M-1,2N-1}} & x_{2M-1,2N} & \cdots \\
x_{2M,0} & x_{2M,1} & \cdots & x_{2M,N-1} & x_{2M,N} & \cdots & x_{2M,2N-1} & x_{2M,2N} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}.$$

Theorem 4. *Suppose that* $\mathrm{ord}_2 X = (N, M)$, *then the elements constituting* $\mathrm{El}_{2N\times 2M}(X)$ *are sufficient to determine expression* (22).

Proof. Proof arises from (27) and (28). To find the column characteristic roots $\mu_1, \ldots, \mu_m$ a maximum of $2M$ elements from each of the first N columns of X is needed, because Hankel determinants of maximum order M need to be constructed. Similarly, determining row characteristic roots $\lambda_1, \ldots, \lambda_n$ requires at most $2N$ elements from the first M rows.

Regardless of whether (27) or (28) is used to determine $c_{kl}^{(st)}$, coefficients $\xi_{ks}^{(j)}$, $j = 0, \ldots, M-1$ or $\eta_{lt}^{(r)}$, $r = 0, \ldots, N-1$ must be computed. That can be performed using only the first $M \times N$ block of X.

Because the computations outlined above are sufficient to determine the canonical expression of X, it can be concluded that the elements of $\mathrm{El}_{2N\times 2M}(X)$ are sufficient for computing characteristic roots of both rows and columns as well as coefficients $c_{kl}^{(st)}$, thus determining (22). □

Example 5. Suppose $\mathrm{ord}_2 X = (3, 3)$ and the elbow of X is given:

$$\mathrm{El}_{6\times 6}(X) = \begin{bmatrix}
1 & 2 & 1 & 0 & 1 & 2 \\
0 & 1 & 0 & -1 & 0 & 1 \\
-1 & 0 & -1 & -2 & -1 & 0 \\
0 & 1 & 0 & & & \\
1 & 2 & 1 & & & \\
0 & 1 & 0 & & &
\end{bmatrix}. \tag{37}$$

The entire 2-sequence can be reconstructed using (22).

Examining $R_k(X), k = 0, 1, 2$ yields the results: $R_0(X)$ is a 1-LRS of order 3 with characteristic roots $1, i, -i$, all of which have multiplicity 1; $R_1(X)$ is a 1-LRS of order 2 with characteristic roots $i, -i$, both with multiplicity 1; $R_2(X)$ is a 1-LRS of order 3 with characteristic roots $1, i, -i$, all of which have multiplicity 1.

This indicates that the row characteristic roots are $\lambda_1 = 1$, $\lambda_2 = i, \lambda_3 = -i$, with multiplicities $n_1 = n_2 = n_3 = 1$. Also, since $n_1 + n_2 + n_3 = 3 = N$, the first three rows are sufficient to determine all row characteristic roots.

Repeating the same procedure with $C_l(X)$, $l = 0, 1, 2$ it is obtained that: the column characteristic roots are $\mu_1 = i$, $\mu_2 = -i$, $\mu_3 = 1$ with $m_1 = m_2 = m_3 = 1$.

By procedures illustrated in previous examples, using only the first 3×3 of X, the coefficients $c_{kl}^{(st)}$ are obtained:

$$C = \begin{bmatrix} c_{11}^{(00)} & c_{21}^{(00)} & c_{31}^{(00)} \\ c_{12}^{(00)} & c_{22}^{(00)} & c_{32}^{(00)} \\ c_{13}^{(00)} & c_{23}^{(00)} & c_{33}^{(00)} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & -\frac{i}{2} & \frac{i}{2} \end{bmatrix}. \tag{38}$$

And the canonical expression reads:

$$x_{jr} = \frac{1}{2} i^j + \frac{1}{2}(-i)^j - \frac{i}{2} i^r + \frac{i}{2}(-i)^r. \tag{39}$$

4. Computational Experiments

It is clear that any real world time series does not have a finite LRS-order simply because real world time series are inevitably contaminated by noise. Otherwise, (if the LRS-order of a real world time series would be finite) the dynamics of the sequence would be governed by a deterministic law — which contradicts to the definition of noise [33].

4.1. *Pseudo-order of a 1-sequence*

A computational framework for the determination LRS pseudo-orders based on the SVD of the Hankel matrix is presented in [15].

Because the computation of the Hankel determinants (11) is numerically unstable, it is unfeasible to use definition of 2-LRS directly to determine if a given 2-sequence has a finite order. To provide a more stable evaluation of a 2-sequence's order, the concept of the pseudo-order is used.

For a 1D sequence $(p_j; j \in \mathbb{Z}_0)$, the pseudo-order is computed using the SVD by the following algorithm [15]:

(1) A Hankel matrix H_K is formed from the sequence $(p_j; j \in \mathbb{Z}_0)$ using the first K elements.

(2) The SVD of H_K is performed:

$$H_K = USV^T, \tag{40}$$

where U, V are the matrices of the orthonormal eigenvectors of $HH^T, H^T H$ respectively and S is a diagonal matrix containing the ordered singular values: $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_K^2 \geq 0$.

(3) For a chosen $\varepsilon > 0$, define the pseudo-order $\widetilde{K}$ of the given sequence as the number of singular values that are greater than ε:

$$\widetilde{K} : \sigma_{\widetilde{K}}^2 > \varepsilon, \quad \sigma_{\widetilde{K}+1}^2 \leq \varepsilon. \tag{41}$$

It is demonstrated in [15] that the pseudo-order of a sequence tends to the true order as $\varepsilon \to 0$, however, setting $\varepsilon = 0$ would lead to a great sensitivity to noise in the sequence $(p_j; j \in \mathbb{Z}_0)$. Thus, for real-world applications it is recommended to choose $\varepsilon > 0$ and investigate the pseudo-order. A number of algorithms that use the concept of 1-LRS have successfully applied this approach [15–17].

It is clear that an alternative approach is also required for the computation of 2-LRS orders.

4.2. *Pseudo-order of a 2-sequence*

The concept of the pseudo-order outlined in the previous section cannot be applied directly to 2-sequences, because they consist of two sets that contain infinitely many 1D sequences. However, the problems that occur in the 1D case are magnified when considering 2-sequences. In particular, it is not immediately clear how to evaluate the row and column orders of a given real-world 2-sequence, because the characteristic roots of each row (column) are influenced by noise.

We propose the following approach to solve this problem using the mean order of the rows (columns) of the given 2-sequence. Let $X = [x_{jr}]_{j,r=0}^{+\infty}$ be a 2-LRS that is homogenous. This means that the differences between the 1D orders of rows (columns) are not large. Suppose the row order of X is equal to N. That means that there exists a set of characteristic roots $\lambda_1, \ldots, \lambda_n$ with multiplicities $n_1, \ldots, n_n$ such that $\sum_{k=1}^{n} n_k = N$.

Then each row of X can be written in the canonical form:

$$x_{jr} = \sum_{k \in \mathcal{I}_j} \sum_{s=0}^{n_k^{(j)}-1} \xi_{ks}^{(j)} \binom{r}{s} \lambda_k^{r-s}, \tag{42}$$

where $j \in \mathbb{Z}_0$; $\mathcal{I}_j \subseteq \{1, \ldots, n\}$ and $n_k^{(j)} \leq n_k$ for all $k \in \mathcal{I}_j$.

Denote $N_j := \sum_{k \in \mathcal{I}_j} n_k^{(j)}$. Then

$$\operatorname{order}(x_{jr}; r \in \mathbb{Z}_0) = N_j. \tag{43}$$

The limit of the mean order of all rows is considered:

$$\overline{N} := \lim_{j \to +\infty} \frac{1}{j} \sum_{k=0}^{j-1} N_k. \tag{44}$$

Since $0 \leq N_k \leq N$, $k \in \mathbb{Z}_0$, the mean order (44) can be written as:

$$\overline{N} = \lim_{j \to +\infty} \frac{1}{j} \sum_{k=0}^{j-1} (N - q_k), \tag{45}$$

where $0 \leq q_k \leq N$. Equation (45) then yields:

$$N - \max_{j \in \mathbb{Z}_0} q_j \leq \overline{N} \leq N - \min_{j \in \mathbb{Z}_0} q_j, \tag{46}$$

which can be rearranged into:

$$\overline{N} + \min_{j \in \mathbb{Z}_0} q_j \leq N \leq \overline{N} + \max_{j \in \mathbb{Z}_0} q_j. \tag{47}$$

If the considered 2-sequence is homogenous, the differences in order between rows q_j are small, thus (47) yields the approximation:

$$\overline{N} \approx N. \tag{48}$$

Using (48) and the SVD, the row pseudo-order of a 2-sequence X can be evaluated by using the algorithm given in Sec. 4.1 with the same ε on each row and considering the mean value of the pseudo-ranks obtained. Thus, the pseudo row rank $\widetilde{N}$ of a homogenous 2-sequence X computed from the first m rows is defined as:

$$\widetilde{N} := \frac{1}{m} \sum_{j=0}^{m-1} \widetilde{N}_j. \tag{49}$$

Analogous computations can also be performed for the columns of a given homogenous 2-sequence X.

4.3. *A synthetic numerical example*

Let us consider two digital images — a black and white image of bricks (Fig. 1(e), denoted as image B) and a grayscale image of uniformly distributed random pixels (Fig. 1(a), denoted as image N). Let us construct a sequence of digital images by assuming discrete values of parameter λ in the following equation:

$$I(\lambda) = (1 - \lambda)N + \lambda B; \quad 0 \leq \lambda \leq 1. \tag{50}$$

It is clear that $I(0) = N, I(1) = B$. The image of bricks evolves from the noise as λ varies from 0 to 1 (Fig. 1).

4.3.1. *LRS pseudo order and Shanon entropy*

It is well known that Shanon entropy $H(X)$ of a digital image determines the randomness of that image [2]. We use standard techniques for the computation of the entropy:

$$H(X) = -\sum_{k=1}^{m} p_k \log_2 p_k, \tag{51}$$

where p_k is the histogram count for the kth of m bins of the given digital image.

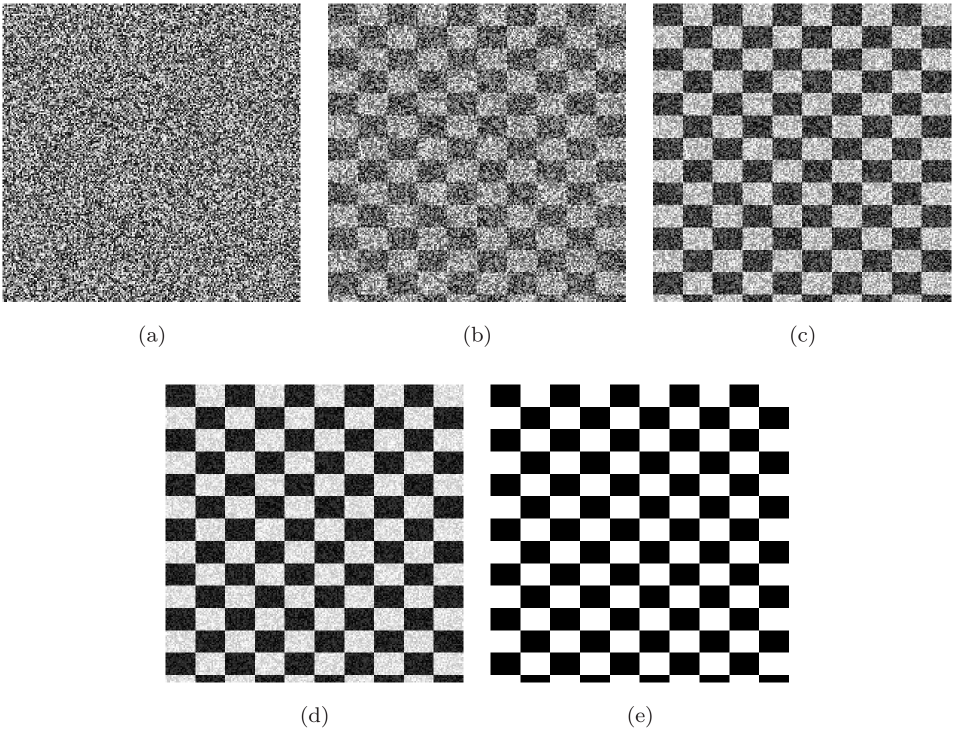


Fig. 1. The digital image of bricks evolves from noise as parameter λ varies from 0 to 1. Digital images in parts (a)–(e) are shown at $\lambda = 0, 0.25, 0.5, 0.75$ and 1 accordingly.

The entropy (51) is computed for a series of digital images as λ is varied from 0 to 1 according to Eq. (50) (Fig. 2). Note that we visualize the inverse of the entropy scaled to the interval $[0, 1]$ in Fig. 2. Such representation helps to clearly interpret the randomness of the evolving image. Entropy is maximal at $\lambda = 0$ and monotonically decreases as the image of bricks becomes clearer (Fig. 2).

There exists a natural connection between the LRS order of a sequence and the algebraic complexity of that sequence [33]. Therefore, one could expect a similar relationship between the 2-LRS order and the complexity of the digital image as well.

We use the same series of digital images represented by Eq. (50) and compute the averaged LRS-pseudo-orders for rows and columns using the algorithm described in Sec. 4.2 (the dimension of the Hankel matrix is set to 80; ε is set to 0.5). However, since the inverse of the entropy is visualized in Fig. 2, we also visualize inverse pseudo-orders in Fig. 2.

The 2-LRS pseudo-order for the image of noise is equal to $(80, 80)$ — corresponding to $(\frac{1}{80}, \frac{1}{80})$ in Fig. 2 at $\lambda = 0$. Then, LRS pseudo-orders for rows and columns monotonically decrease as λ is varied from 0 to 1 (Fig. 2). However, the variation of LRS pseudo-orders for rows and columns is not identical. The periodicity of the

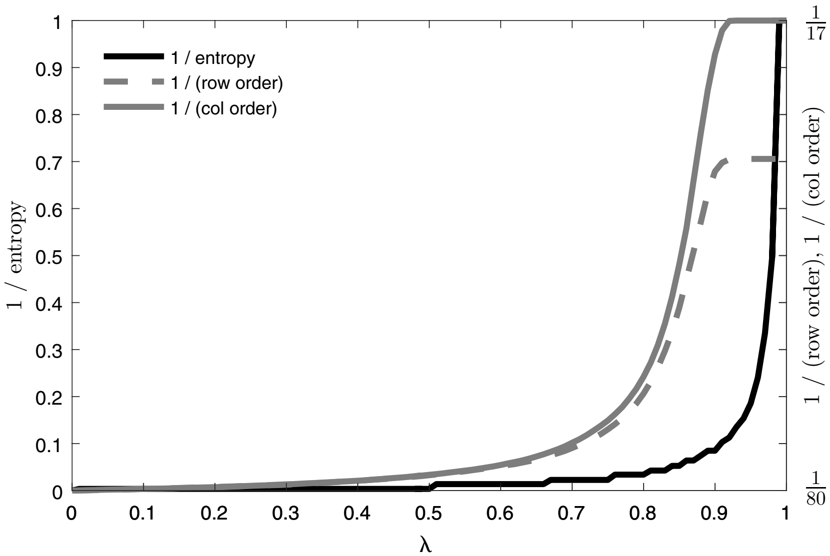


Fig. 2. Evolution of entropy and row/column LRS pseudo-order as parameter λ varies from 0 to 1. The left y-axis represents the inverse of the entropy $H(X)$. The right y-axis represents the inverse of the average row and column LRS-pseudo-order.

image of bricks along the rows is longer compared to the periodicity of this image along the columns (this is due to the shape of the bricks). The shorter average period results into a smaller LRS pseudo-order. 2-LRS pseudo-order for the image of bricks (without noise) is equal to $(22, 17)$ corresponding to $(\frac{1}{22}, \frac{1}{17})$ in Fig. 2 at $\lambda = 1$.

Therefore, the variation of 2-LRS pseudo-order of the digital image $I(\lambda)$ reveals not only the evolution of the complexity of the image — but also the geometrical orientation of the evolving pattern.

4.3.2. *LRS pseudo order and image correlation*

Shanon entropy is a general measure of image randomness, thus it cannot be used to measure randomness horizontally (along the rows) or vertically (along the columns) in a given image. To perform measurements of randomness in horizontal and vertical directions, we use correlation, one of the Haralick features derived from co-occurrence matrices [9]:

$$\rho_H(X) := \frac{1}{\sigma_x \sigma_y} \left(\sum_{i=1}^{N_g} \sum_{j=1}^{N_g} (ij) p(i,j) - \mu_x \mu_y \right), \tag{52}$$

where N_g is the number of gray levels in the image X; $p(i,j)$ is the entry of the co-occurrence matrix (the probability that the pixel with gray level i is adjacent to the pixel with gray level j); $\mu_x, \sigma_x, \mu_y, \sigma_y$ are the means and standard deviations of the partial probability density functions of the co-occurrence matrices.

Horizontal and vertical adjacency is used to compute two correlations for each image using (52); the correlation computed using horizontal and vertical adjacency is referred to as row and column correlations, respectively.

A comparison of row and column LRS pseudo orders and correlations of the image sequence of bricks (Fig. 1) is pictured in Fig. 3. As λ is varied from 0 to 1, both row and column correlations increase — while row and column orders decrease monotonically. As noted previously, the periodicity of the image sequence of bricks in Fig. 1 is not equal along the rows and columns. This effect can be explained because the period is longer along the rows. Thus, the LRS pseudo order of rows is larger compared to the pseudo order of columns. The LRS pseudo order for rows and columns is 22 and 17 respectively at $\lambda = 1$. A similar effect is observed with respect to the Haralick feature of correlation — row correlation is higher than column correlation because more adjacent pixels are of the same gray level. Row correlation is almost equal to 1 when $\lambda > 0.9$ and column correlation is 0.82 in the same range.

This computational experiment demonstrates that LRS pseudo-orders do represent the evolution of complexity in digital images along the horizontal and vertical axis.

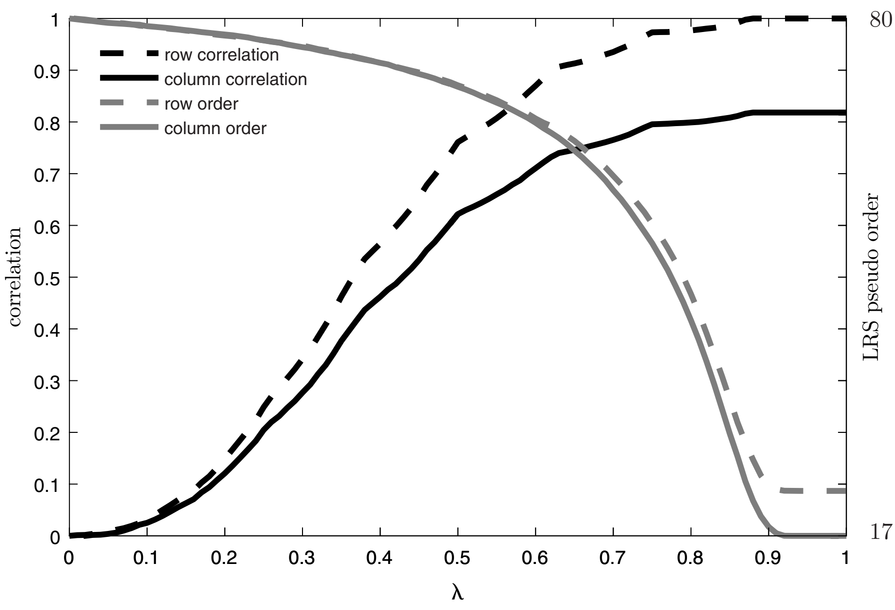


Fig. 3. Evolution of row/column correlation and row/column LRS pseudo-order as parameter λ varies from 0 to 1. The left y-axis represents the row/column correlations $\rho_H(X)$. The right y-axis represents the row/column LRS-pseudo-order.

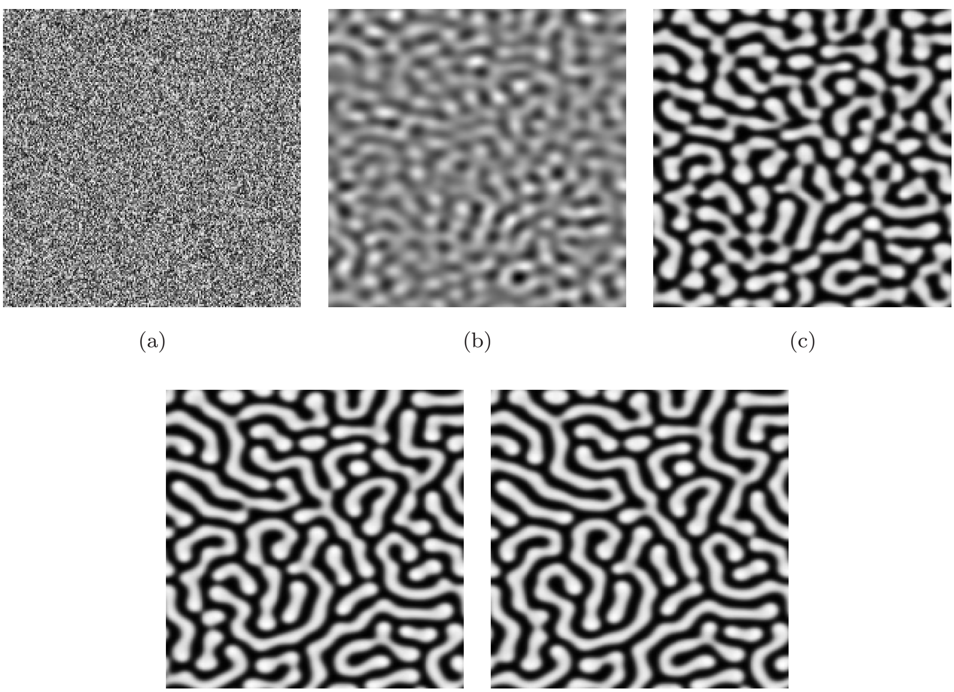
4.4. 2-sequence pseudo order of self-organizing patterns

We will use Beddington–de-Angelis-type predator-prey model with self- and cross-diffusion [35, 40]:

$$\frac{\partial N}{\partial t} = r\left(1 - \frac{N}{K}\right) - \frac{\beta N}{B + N + wP}P + D_{11}\nabla^2 N + D_{12}\nabla^2 P; \tag{53}$$

$$\frac{\partial P}{\partial t} = \frac{\epsilon \beta N}{B + N + wP}P - \eta P + D_{21}\nabla^2 N + D_{22}\nabla^2 P, \tag{54}$$

where t is time; N and P are densities of preys and predators; β is a maximum consumption rate; B is a saturation constant; w is a predator interference parameter; η represents per capita predator death rate; and ϵ is the conversion efficiency of food into offspring. Nonzero initial conditions $N(x, y, 0) > 0$; $P(x, y, 0) > 0$ are set in a rectangular domain with periodic boundary conditions. The following set $D_{11} = 0.01$, $D_{12} = 0.0115$, $D_{21} = 0.01$, $D_{22} = 1$, $r = 0.5$, $\epsilon = 1$, $\beta = 0.6$, $K = 2.6$, $w = 0.4$, $B = 0.3154$ results in the evolution of a self-organizing pattern from the equilibrium point $(N^*, P^*) = (0.430580, 0.718555)$ which is perturbed by small random perturbation [35]. Computational reconstruction of the evolution of self-organizing pattern of preys from random initial conditions is illustrated in Fig. 4.



(a) (b) (c)

(d) (e)

Fig. 4. The evolution of self-organizing Beddington–de-Angelis type patters. Digital images in parts (a)–(e) are shown at 0, 15,000, 25,000, 50,000 and 70,000 time forward steps accordingly.

We repeat the computational experiments with a sequence of images representing the evolution of self-organizing patterns (Fig. 5) with ε set to 0.5. Now, the evolution of the entropy is very complex and nonmonotonous (Fig. 5). However, computation of 2-LRS pseudo-orders reveal the hidden rules of the complexity variation during the evolution of the self-organizing pattern.

Initially, the image is random — so 2-LRS pseudo-order for the image of noise is equal to (80, 80) (Fig. 5). Then, the self-organizing pattern starts to evolve and the complexity of the image decreases — 2-LRS pseudo-order is equal to $(16.6, 13.7)$ at $t = 15{,}000$ (the time step of time forward iteration is 0.01). However, the complexity of image suddenly starts to increase again at $15{,}000 \leq t \leq 25{,}000$. Astonishingly, the complexity of the fully developed pattern is higher compared to the complexity of pattern in the middle stage of development (2-LRS pseudo-order is equal to $(24.2, 21.2)$ at $t = 70{,}000$).

Such an effect can be explained by a rather simple (though not trivial) consideration. The fully developed pattern is not a regular pattern. The distribution of stripes (and the forms of stripes) in the fully developed image are governed by a large scale spatial chaos law. Note that this pattern is unique for every initial condition — different random initial conditions result into different patterns of stripes.

Initial random conditions could be considered as small scale spatial chaos in that respect. However, it is interesting to observe that the evolution from small scale spatial chaos to large scale spatial chaos is not straightforward. First, random initial conditions evolve into a seemingly regular pattern of spatial waves. However, Turing instability [23] deforms these almost regular waves into a complex irregular pattern of large scale stripes. 2-LRS pseudo-orders allow efficient and clear visualization of these complex processes of transformation.

Nonmonotonous effects are observed in the evolution of the row and column correlation (Fig. 6). Both row and column correlations reach a peak value of almost 1 at $t = 20{,}000$. After this peak, both correlations dip slightly but do not fluctuate: they maintain values above 0.98 in the interval $20{,}000 < t \leq 70{,}000$. Note that the values of row and column correlations do not differ significantly one from another during the evolution of the image. This situation is completely different for row and column pseudo orders — they do separate one from another. This feature enables to draw conclusions about the complexity of the digital image in the horizontal and vertical directions.

Moreover, 2-LRS pseudo-orders exemplify the orientation of stripes in self-organized patterns. The bricks are elongated along the horizontal axis in Fig. 1. Thus, the period along the rows is longer and the mean LRS-order of the rows is larger compared to the columns (Fig. 2). The same effect can be observed for self-organizing patterns (Fig. 4). Figures 5 and 6 demonstrate that the mean row LRS-order is larger compared to the mean column LRS-order. This implies that the pseudo-period along the rows in Fig. 4 is longer compared to the columns.

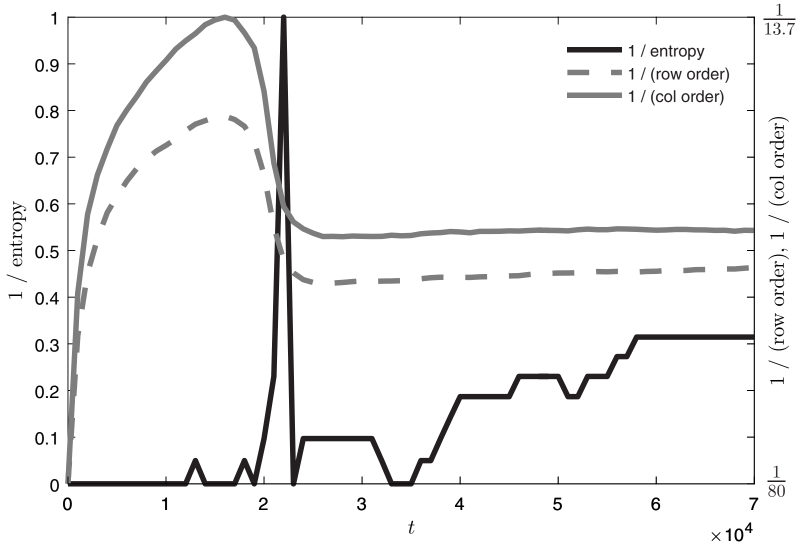


Fig. 5. Evolution of entropy and row/column LRS pseudo-order of the Beddington–de-Angelis type self-organizing pattern for 70,000 time-forward steps. The left y-axis represents the inverse of the entropy $H(X)$. The right y-axis represents the inverse of the average row and column LRS-pseudo-order.

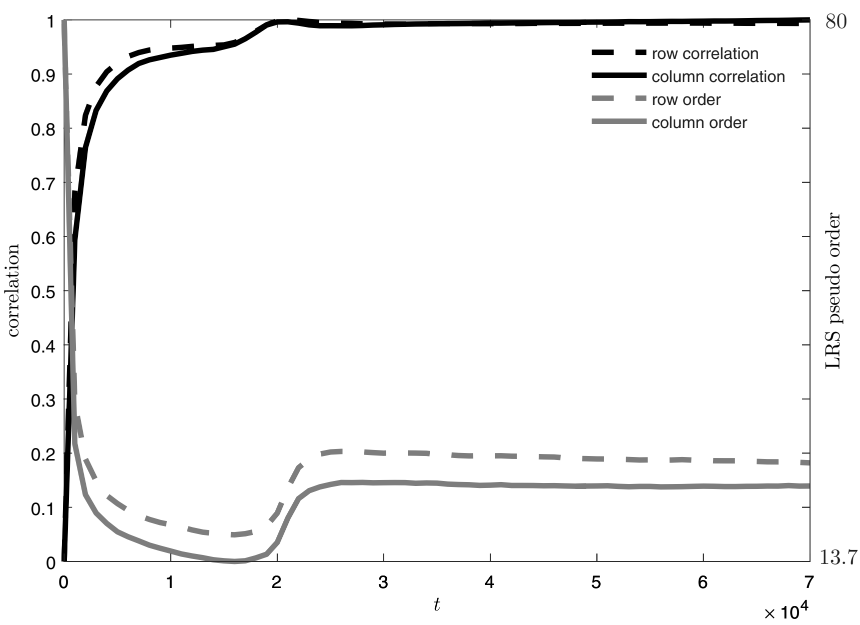


Fig. 6. Evolution of row/column correlation and row/column LRS pseudo-order of the Beddington–de-Angelis type self-organizing pattern for 70,000 time-forward steps. The left y-axis represents the row/column correlations $\rho_H(X)$. The right y-axis represents the average row/column LRS-pseudo-order.

5. Concluding Remarks

The concept of the order of a 2-sequence is presented in this paper. The extension of the order of 1-sequence to the order of 2-sequence is far from being trivial — it requires the introduction of the concept of the minimal information subset of 2-sequence. The canonical expression theorem defines the necessity and sufficiency for the generation of 2-LRS.

It is demonstrated that using the SVD, the concept of 2-LRS can be successfully applied for the analysis of image complexity. Because of the ability to measure complexity along the x- and y-axis, the row and column 2-LRS pseudo-orders provide a deeper insight into the complexity of images compared with Shanon entropy.

2-LRS can also be used to analyze self-organizing patterns. It is shown that unlike Shanon entropy, 2-LRS pseudo-order can be applied to detect the formation of almost regular patterns that evolve from small scale spatial chaos and deform due to Turing instability as time moves forward and large scale spatial chaos appears.

There are a number of potential applications of 2-LRS that are presented in this paper. One of the most intriguing is the construction of an algebraic approximation of any 2D image using the canonical expression of 2-LRS. In the 1D case, approximations constructed using 1-LRS theory have been shown to be effective in reducing the Runge effect and dealing with noise in real-world sequences [13].

New results on the construction of such approximations in the 2D case would enable the use of 2-LRS theory in image compression applications (exploiting the concept of minimal information subsets), image reconstruction and filtering, including denoizing algorithms.

The theoretical and practical results presented in this paper also open new possibilities for applications in the area of complexity analysis, 2D interpolation, information compression and encryption. These applications remain definite objectives of future research.

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