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The solitary solution of the Liouville equation produced by the Exp-function method does not hold for all initial conditions

Minvydas Ragulskis [a,*], Zenonas Navickas [b], Liepa Bikulciene [b]

[a] Research Group for Mathematical and Numerical Analysis of Dynamical Systems, Kaunas University of Technology, Studentu 50-222, Kaunas LT-51368, Lithuania
[b] Department of Applied Mathematics, Kaunas University of Technology, Studentu 50-325, Kaunas LT-51368, Lithuania

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ABSTRACT

In this paper we argue that the solitary solution of the Liouville equation produced by the Exp-function method does not satisfy the original differential equation for all initial conditions. Moreover, the region where the solution is correct is located entirely on a curve in the parameter plane of initial conditions. We derive the explicit equation for this curve and argue that classical Exp-function type methods cannot identify such constraints related to initial conditions.

1. Introduction

We consider the Liouville equation

$$\frac{\partial^2 u}{\partial \xi \partial \tau} = -\exp(u). \quad (1)$$

Using the transformation $y = \exp(u)$ and $x = k\xi + \Omega\tau + \varphi$, Eq. (1) becomes [1]

$$k\Omega yy'' - k\Omega (y')^2 + y^3 = 0, \quad (2)$$

where the prime denotes the derivative with respect to x. Let $k\Omega = \frac{1}{2\gamma}$; Eq. (2) turns out to be

$$yy'' - (y')^2 + 2\gamma y^3 = 0; \quad \gamma \neq 0, \quad (3)$$

where γ is a parameter of the differential equation.

We will seek a partial solution $y = y(x)$ which satisfies initial conditions $y(0) = s$ and $y'_x(0) = t$. Moreover, we will seek only such solutions as can be expressed as a ratio of finite sums of exponential functions. Our motives stem from the fact that Wu and He [1] propose the structure of the solution and then show that

$$y = \frac{a_1 \exp(x) + a_0 + a_{-1}\exp(-x)}{\exp(x) + b_0 + b_{-1}\exp(-x)} \quad (4)$$

is a solution of Eq. (2) where the scalar coefficients are found using ordinary symbolic computations.

* Corresponding author. Tel.: +370 69822456; fax: +370 37330446.
E-mail address: minvydas.ragulskis@ktu.lt (M. Ragulskis).
URL: http://www.personalas.ktu.lt/~mragul (M. Ragulskis).

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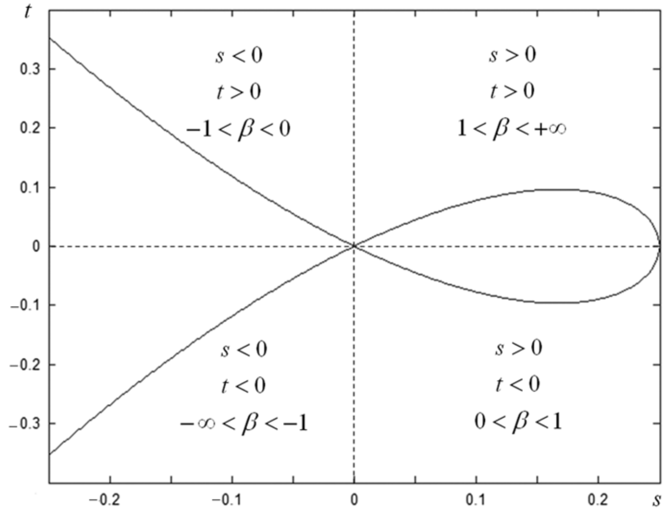

Fig. 1. The graphical representation of the constraint linking initial conditions s and t at $\gamma = 1$.

The objective of this paper is to show that Eq. (4) is a solution of the differential equation (2) for only a small region of initial conditions in the parameter plane $(s; t)$. Moreover, we will show that this region is a curve in the parameter plane of initial conditions.

2. Finding, but not guessing, the partial solution

As mentioned previously, Exp-function type methods are based on the proposition that the solution of an ordinary differential equation has a certain form. We will use a technique presented in [2] and will derive a partial solution of Eq. (2) which can be explicitly expressed as a ratio of finite sums of exponential functions without guessing how many terms there are in the numerator or the denominator.

We perform the variable change $z = \exp(x)$ and construct the image differential equation [2] for Eq. (3):

$$\omega(z^2\omega''_{zz} + z\omega'_z) - z^2(\omega'_z)^2 + 2\gamma\omega^3 = 0, \tag{5}$$

where $\omega = \omega(z)$; $\omega(1) = s$ and $\omega'_z(1) = t$. The generalized differential operator [2] of the image differential equation reads

$$D = D_c + tD_s + \frac{c^2t^2 - cst - 2\gamma s^3}{c^2 s} \cdot D_t. \tag{6}$$

Then, the exact solution of Eq. (5) reads $\omega(z) = \sum_{j=0}^{+\infty} \frac{(z-\exp(0))^j}{j!} \cdot D^j s = \sum_{j=0}^{+\infty} (z-1)^j \hat{p}_j$ [2] where $\hat{p}_j = \frac{1}{j!} D^j s$; $(\hat{p}_j; j = 0, 1, 2, \ldots)$ and $y(x) = \omega(\exp(x))$. Elementary computations yield the sequence of determinants of the Hankel matrices [2]:

$$\begin{aligned}
&\det H_1 = |\hat{p}_0| = s;\\
&\det H_2 = \begin{vmatrix} \hat{p}_0 & \hat{p}_1 \\ \hat{p}_1 & \hat{p}_2 \end{vmatrix} = -\frac{c^2t^2 + sct + 2\gamma s^3}{2c^2};\\
&\det H_3 = -\frac{c^2t^2 - s^2 + 4\gamma s^3}{144 s^3 c^6}(c^4t^4 + 3sc^3t^3 + 4c^2t^2s^3 + 2t^2s^2c^2 + 6ts^4c - 12s^6);\\
&\det H_4 = \frac{(c^2t^2 - s^2 + 4\gamma s^3)^2}{1036800\, s^8 c^{12}}(c^8t^8 + 6c^7st^7 - 18c^5s^3t^5 + 456c^3s^7t^3\gamma^2 + 1440cs^{10}t\gamma^3 + 42c^4s^5t^4\gamma + 10c^6s^3t^6\gamma\\
&\quad + 48c^5s^4t^5\gamma^2 - 88c^2s^7t^2\gamma + 360cs^9t\gamma^2 - 72c^3s^6t^3\gamma + 1112c^2s^9t^2\gamma^3 + 236c^2s^8t^2\gamma^2 + 212c^4s^6t^4\gamma^2\\
&\quad - 24c^3s^5t^3 + 7c^6s^2t^6 - 44c^4s^4t^5 + 360s^{11}\gamma^3 + 1440s^{12}\gamma^4).
\end{aligned} \tag{7}$$

Thus, $\det H_3|_{c=1} = \det H_4|_{c=1} = 0$ ($c = 1$ because $\exp(0) = 1$) when

$$t^2 - s^2 + 4\gamma s^3 = 0, \tag{8}$$

or $t = \pm s\sqrt{1 - 4\gamma s}$; $s \le \frac{1}{4\gamma}$. The graph of the curve (8) is illustrated in Fig. 1.

Thus, the solution of the differential equation (2) can be expressed as a ratio of finite sums of exponential functions when the variables of initial conditions s and t satisfy equality (8).

Now the algorithm described in [2] can be used to construct the function $\omega(z)$. It is sufficient to check whether this function satisfies the original differential equation for concluding that it is a partial solution.

First of all, we analyze the arc of the curve (8) in the first quarter of the parameter plane $(s; t)$ where $s > 0$ and $t \geq 0$ (Fig. 1).

The Hankel characteristic equation [2] takes the following form:

$$\det \left. \begin{bmatrix} s & Ds & \frac{1}{2}D^{2s} \\ Ds & \frac{1}{2}D^2 s & \frac{1}{6}D^{3s} \\ 1 & \rho & \rho^2 \end{bmatrix} \right|_{\substack{c=1 \\ t=s\sqrt{1-4\gamma s}}} = 0, \tag{9}$$

which yields $\rho_1 = \rho_2 = \frac{1}{2}(-1 + \sqrt{1-4\gamma s}) := \rho$, where $\sqrt{1-4\gamma s}$ is an arithmetic nonnegative root from the nonnegative number $(1-4\gamma s)$. Next, a system of linear algebraic equations for the determination of coefficients μ_1 and μ_2 is constructed [2]:

$$\left. \frac{D^j s}{j!} \right|_{\substack{c=1 \\ t=s\sqrt{1-4\gamma s}}} = \mu_1 \rho^j + \mu_2 \binom{j}{1} \rho^{j-1}; \quad j = 0, 1, 2, \ldots. \tag{10}$$

Eq. (10) is equivalent to

$$\left(\frac{1}{2}(\sqrt{1-4\gamma s} - 1)\right)^j \mu_1 + \left(\frac{1}{2}(\sqrt{1-4\gamma s} - 1)\right)^{j-1} \cdot j \cdot \mu_2 = \left. \frac{D^j s}{j!} \right|_{\substack{c=1 \\ t=s\sqrt{1-4\gamma s}}}; \quad j = 0, 1. \tag{11}$$

Elementary computations yield $\mu_1 = s$ and $\mu_2 = \frac{s}{2}(1 + \sqrt{1-4\gamma s})$. Finally,

$$\begin{aligned} \omega(z)|_{\substack{c=1 \\ t=s\sqrt{1-4\gamma s}}} &= \sum_{j=0}^{+\infty} (z-1)^j \left(s\left(\frac{1}{2}(\sqrt{1-4\gamma s} - 1)\right)^j + \frac{s}{2}(1 + \sqrt{1-4\gamma s}) j \left(\frac{1}{2}(\sqrt{1-4\gamma s} - 1)\right)^{j-1} \right) \\ &= \frac{2s(1 - 2\gamma s + \sqrt{1-4\gamma s})z}{(2\gamma s z + 1 - 2s + \sqrt{1-4\gamma s})^2} = \frac{\frac{1-2\gamma s+\sqrt{1-4\gamma s}}{2\gamma^2 s} \cdot z}{\left(z + \frac{1-2\gamma s+\sqrt{1-4\gamma s}}{2\gamma s}\right)^2}. \end{aligned} \tag{12}$$

Further computations can be simplified by introducing a parameter $\beta := \frac{1-2\gamma s+\sqrt{1-4\gamma s}}{2\gamma^2 s}$ (when $s > 0$ and $t \geq 0$):

$$\omega(z)|_{\substack{c=1 \\ t=s\sqrt{1-4\gamma s}}} = \frac{\beta \cdot z}{(z + \gamma\beta)^2}. \tag{13}$$

Analogous computations of ρ_1, ρ_2, μ_1 and μ_2 are performed for the remaining quarters of the parameter plane $(s; t)$ (Fig. 1). It is easy to show that

$$\bar{\omega}(z) := \omega(z)|_{\substack{c=1 \\ t^2-s^2+4\gamma s^3=0}} = \frac{\beta \cdot z}{(z + \gamma\beta)^2} \tag{14}$$

where

$$\beta := \frac{1 - 2\gamma s + \mathrm{sgn}(st)\sqrt{1-4\gamma s}}{2\gamma^2 s}; \quad s \neq 0; \qquad \mathrm{sgn} x := \begin{cases} 1, & \text{when } x > 0; \\ 0, & \text{when } x = 0; \\ -1, & \text{when } x < 0. \end{cases} \tag{15}$$

It is easy to check that the function $\bar{\omega}(z) := \frac{\beta \cdot z}{(z+\gamma\beta)^2}$ satisfies differential equation (2). Therefore,

$$y(x) = \bar{\omega}(\exp(x)) = \frac{\beta \exp(x)}{(\exp(x) + \gamma\beta)^2} \tag{16}$$

is a partial solution of the differential equation (3) satisfying initial conditions $y(0) = \frac{\beta}{(\gamma\beta+1)^2}$ and $y'_x(0) = \frac{\beta(\gamma\beta-1)}{(\gamma\beta+1)^3}$ for all β (except $\beta = -1$).

Thus, according to [3], $\det H_k|_{\substack{c=1 \\ t^2-s^2+4\gamma s^3}} = 0 = 0$; $k = 3, 4, \ldots$; $Hr(\frac{1}{j!}D^j s|_{\substack{c=1 \\ t^2-s^2+4\gamma s^3=0}}; j = 0, 1, 2, \ldots) = 2$ and equalities (10) hold true. It can be noted that the solution exists at $s \leq \frac{1}{4\gamma}$ when $\gamma > 0$ and at $s \geq \frac{1}{4\gamma}$ when $\gamma < 0$. A graphical representation of constraints linking the initial conditions s and t at different values of γ is given in Fig. 2. It can be noted that Eq. (16) coincides with the solution presented in [1] (divide the numerator and denominator of Eq. (16) by $\exp(x)$). But we would like to stress that we did not guess the structure of the solution; it has been automatically identified by the direct application of the algorithm described in [2]. This is the main difference between classical Exp-function type methods and our proposed technique.

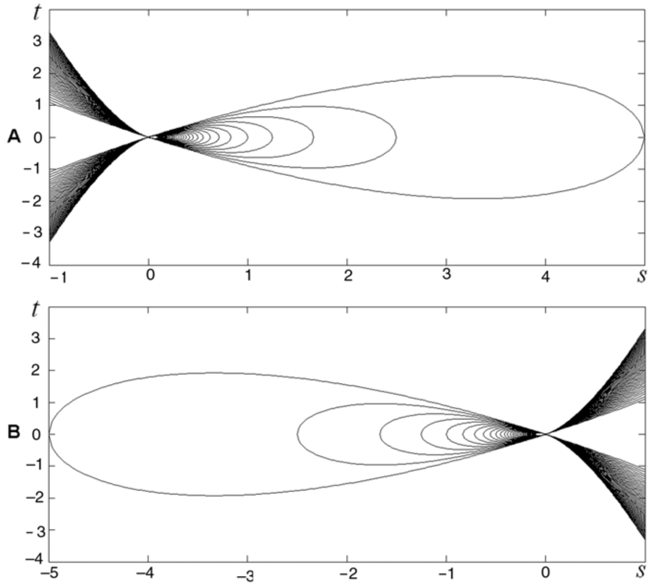

Fig. 2. A graphical representation of constraints linking initial conditions s and t at: (A) $\gamma = 0.05 + 0.05(j-1)$; $j = 1, 2, \ldots, 50$; and (B) $\gamma = -0.05 - 0.05(j-1)$; $j = 1, 2, \ldots, 50$.

But an even more important fact is that our approach enabled to show that the solution of the differential equation (2) takes the form (16) only when the initial conditions satisfy Eq. (8). This is an important fact, and we argue that the partial solution cannot be expressed in the form represented by Eq. (16) for all initial conditions. Our technique enabled the identification of the constraint linking two initial conditions. Classical Exp-function type methods are incapable of finding such constraints and may produce wrong results in general, which is clearly illustrated by this analysis.

3. Special solutions

So far, we have analyzed partial solutions of differential equation (3) for when $\gamma \neq 0$. It follows from the relationship (8) that we omit lines $|s| = |t|$ from the phase plane of initial conditions. That is an important argument. Eq. (16) is not a solution of the differential equation (3) if just $|s| = |t|$. These questions are not discussed at all in [1].

First we will analyze solutions on the line $t = -s$. From Eq. (9) it follows that $\rho_1 = \rho_2 = 0$ and from Eq. (11) it follows that $\mu_1 = \mu_2 = s$. Then,

$$\omega_1(z)|_{\substack{c=1\\t=-s}} = \sum_{j=0}^{+\infty} \frac{(z-1)^j}{j!} \cdot j! \left(s \cdot 0^j + s \binom{j}{1} 0^{j-1} \right) = s + s(z-1) = sz \tag{17}$$

where $0^0 = 1$; $\binom{0}{1} 0^{-1} = 0$.

Analogously, when $t = s$, $\rho_1 = \rho_2 = -1$ and $\mu_1 = s$; $\mu_2 = 0$. Then,

$$\omega_2(z)|_{\substack{c=1\\t=s}} = \sum_{j=0}^{+\infty} \frac{(z-1)^j}{j!} \cdot j! \left(s(-1)^j + 0 \cdot \binom{j}{1} (-1)^{j-1} \right) = s \sum_{j=0}^{+\infty} (-1)^j (z-1)^j = s \frac{1}{1+(z-1)} = \frac{s}{z}. \tag{18}$$

It can be noted that both solutions $\omega_1(z) = sz$ at $\omega(1) = s$; $\omega_z'(1) = -s$, and $\omega_2(z) = \frac{s}{z}$ at $\omega(1) = \omega_z'(1) = s$ do satisfy the differential equation (5) (in both cases $\gamma = 0$). Furthermore, $Hr\left(\frac{D^j s}{j!}|_{\substack{c=1\\t=-s}}; j = 0, 1, 2, \ldots\right) = Hr(s, s, 0, 0, \ldots) = 2$; but $Hr\left(\frac{D^j s}{j!}|_{\substack{c=1\\t=s}}; j = 0, 1, 2, \ldots\right) = Hr(s, -s, s, -s, \ldots) = 1$. Also, $\beta = s$ at $\gamma = 0$ and $t = -s$, but $\beta = \pm\infty$ at $\gamma = 0$ and $t = s$. It can be noted that the same results can be derived by solving the following differential equation:

$$\omega(z^2 \omega_{zz}'' + z\omega_z') - z^2 (\omega_z')^2 = 0 \quad \text{at } \omega(1) = s; \qquad \omega_z'(1) = \pm s. \tag{19}$$

It is clear that the solutions presented in Eq. (17) cannot be expressed as a ratio of finite sums of exponential functions — their structure is much simpler.

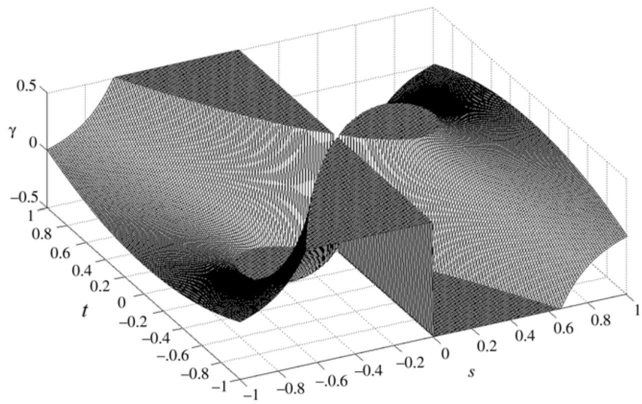

Fig. 3. A graphical representation of the surface $-0.5 \le \gamma(s,t) \le 0.5$.

4. The relationships of the initial conditions and the parameters of the differential equation and its solutions

Let us introduce the following notation:

$$a = \beta; \quad b = \gamma\beta. \tag{20}$$

Then, the solution of differential equation (3) is

$$y = \frac{a\exp(x)}{(\exp(x)+b)^2}; \qquad y(0) = s; \qquad y'_x(0) = t. \tag{21}$$

Then, the following relationships hold true:

$$\begin{cases} s = \dfrac{a}{(b+1)^2}; \\ t = \dfrac{a(b-1)}{(b+1)^3}; \end{cases} \quad b \ne -1; \tag{22}$$

and

$$\begin{cases} a = \dfrac{4s^3}{(s-t)^2}; \\ b = \dfrac{s+t}{s-t}; \end{cases} \quad s \ne t. \tag{23}$$

Moreover,

$$\gamma = \frac{s^2-t^2}{4s^3}. \tag{24}$$

In other words, given any initial conditions (except $s = 0$), one can find such a γ when Eq. (21) is the solution of the differential equation (3). By the way, this solution holds for all other initial conditions located on a level isoline of the surface $\gamma(s,t)$ (Fig. 3). Besides mathematical aesthetics, this relationship also has an important physical meaning. A unique circular frequency Ω corresponds to a given wavenumber k at preset initial conditions. Different initial conditions (on a different isoline) would result in another circular frequency Ω for the same wavenumber k.

Relationships (22)–(24) may also help to explain the essence of the Exp-function method. One can derive necessary relationships (22) and (23) given a general explicit second-order ordinary differential equation $y'' = P(x, y, y')$ with initial conditions $y(0) = s$; $y'(0) = t$ and having a solution in the form of Eq. (21). The relationship (24) can be derived only when we have a concrete differential equation (3) and the solution (21).

But such an approach has one major drawback. One must virtually guess the solution of a nonlinear differential equation. The Exp-function method is basically a symbolic computation technique which in fact tunes the parameters of the solution which has been guessed beforehand. Why should one try guessing solutions when the operator method not only generates the structure of the solution automatically, but also automatically produces the relationship (24)?

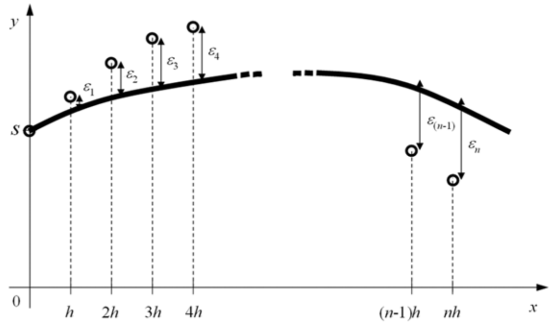

Fig. 4. A schematic diagram illustrating differences between the analytical solution $y(x)$ (the thick solid line) and the approximate computational solution (represented by circles) for the first n steps after initial conditions $y(0) = s$; $y'_x(0) = t$.

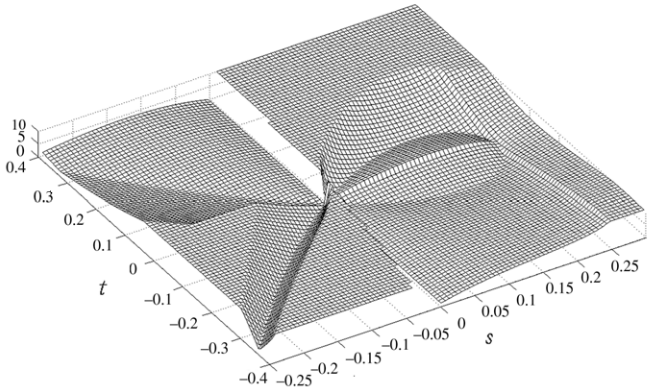

Fig. 5. The distribution of errors between the analytical solution and the computational solution in the parameter plane of initial conditions at $\gamma = 1$.

5. Computational experiments

We will check the validity of the results produced by carrying out a computational experiment. We will solve the initial problem

$$yy'' - (y')^2 + 2\gamma y^3 = 0; \qquad y(0) = s; \qquad y'_x(0) = t \tag{25}$$

at $\gamma = 1$ using approximate computational constant step marching techniques. Let us denote the approximate partial solution as $\tilde{y}(0 + hk)$; $k = 0, 1, 2, \ldots$; $\tilde{y}(0) = s$, where h is the step size. The exact analytical partial solution in Eq. (16) is defined on Eq. (8), but we release the constraint (8) and assume that the solution (16) is valid throughout the space of initial conditions. We travel 100 steps from the predefined initial conditions and compute differences between the approximate computational solution and the exact analytical "solution" defined by Eq. (16) (Fig. 4). Adding absolute differences for 100 steps produces an error estimate:

$$\varepsilon(s, t) = \sum_{k=1}^{100} |\varepsilon_k| = \sum_{k=1}^{100} \left| \tilde{y}(0 + hk) - \frac{\beta \exp(0 + hk)}{(\exp(0 + hk) + \beta)^2} \right|, \tag{26}$$

where the parameter β is defined by Eq. (16). The distribution of $\varepsilon(s, t)$ is illustrated in Fig. 5. Numerical values of $\varepsilon(s, t)$ higher than 10 are truncated to 10 in order to make the figure more comprehensive. It can be clearly seen that the errors are almost equal to zero on the curve defined by Eq. (8).

6. Concluding remarks

We have shown that the solitary solution of the Liouville equation produced by the Exp-function method does not satisfy the original differential equation for all initial conditions. We have used an alternative operator based method to derive the solitary solution of the Liouville equation and have identified the region in the parameter plane of the initial conditions for which this solution does exist. We argue that it is impossible to find such regions using classical Exp-function type methods.

References

[1] X.H. Wu, J.H. He, Solitary solutions, periodic solutions and compacton-like solutions using the Exp-function method, Computers and Mathematics with Applications 54 (2007) 966–986.
[2] Z. Navickas, M. Ragulskis, How far one can go with the Exp-function method? Applied Mathematics and Computation 211 (2009) 522–530.
[3] Z. Navickas, L. Bikulciene, M. Ragulskis, Generalization of Exp-function and other standard function based methods, Applied Mathematics and Computations 216 (2010) 2380–2393.