COMMUNICATIONS IN NUMERICAL METHODS IN ENGINEERING
*Commun. Numer. Meth. Engng* 2006; **22**:347–356
Published online 21 September 2005 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/cnm.816

# Harmonic balance method for FEM analysis of fluid flow in a vibrating pipe

M. Ragulskis[1,*,†], A. Fedaravicius[2] and K. Ragulskis[3]

[1] *Department of Mathematical Research in Systems, Kaunas University of Technology, Kaunas, Lithuania*
[2] *Department of Mechanical Engineering, Kaunas University of Technology, Kaunas, Lithuania*
[3] *Lithuanian Academy of Sciences, Kaunas, Lithuania*

## SUMMARY

A numerical procedure for the analysis of non-Newtonian fluid flow in a longitudinally vibrating tube is developed. The formulation of the problem is presented in differential equation form and finite element model is developed leading to the first-order matrix differential equation. A special modification of the harmonic balance procedure is proposed for this non-linear problem. Numerical validation of the harmonic balance procedure was performed by comparison of the average mass flow rate with the results of direct time integration. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: finite element method; harmonic balance; fluid flow

## INTRODUCTION

The problem of fluid flow control exploiting the vibrations of the boundary is important in the process of design of various engineering devices and optimization of processes of conveyance [1–4]. Such applications cover the transport of various suspensions, industrial wastes and many types of crude oils, especially at low temperatures. Analysis of such non-linear dynamical systems requires the development of adequate mathematical models and appropriate strategies for numerical modelling [5–8].

Tube vibrations induced by internal or external flow is an important engineering problem [9]. Nevertheless, analysis of an inverse problem—fluid flow control by forced vibrations of the tube itself is also of great interest. Such a vibration-based flow control methodology could enrich existing flow control techniques and provide background data for the development of new types of liquid material dosing equipment. It is assumed that the boundary (the tube) is a non-deformable rigid body.

*Correspondence to: M. Ragulskis, Department of Mathematical Research in Systems, Kaunas University of Technology, Studentu 50-222, Kaunas, LT-51368, Lithuania.
†E-mail: minvydas.ragulskis@ktu.LT

*Received 28 June 2004*
*Revised 5 January 2005*
*Accepted 27 June 2005*

The finite element formulation is used together with the approximate procedure for the analysis of non-linear steady-state vibrations which consists from two basic steps. First a finite element model is developed leading to the first-order matrix differential equation. Then an approximate steady-state solution is sought using the proposed harmonic balance procedure which consists from the following main steps:

(1) problem formulation in differential equation form;
(2) development of the finite element problem formulation;
(3) modal decomposition of the solution;
(4) development and application of the harmonic balance method.

The harmonic balance method in general is one of the most straightforward and practical methods for estimating periodic steady-state solutions and can be applied to non-linear equations with a periodic forcing term, thus providing better understanding of the reasons for the non-linear effects taking place in the system [10].

A semi-analytical method for calculating a non-Newtonian fluid flow in a vibrating pipe is presented. The method uses an efficient harmonic balance approach to evaluate steady-state fluid flow eliminating the time integration.

## FINITE ELEMENT MODEL

A Cartesian coordinate system is defined where the $z$-axis is parallel to the axis of the tube (Figure 1). It is assumed that the cross-section of the tube does not vary with the $z$ coordinate and the velocity of fluid flow does not depend on $z$, i.e. $w = w(x, y, t)$, where $w$ denotes the velocity and $t$ the time. The other components of velocity are assumed to be equal to zero. Thus the condition of incompressibility is satisfied identically. The stresses take the form

$$\sigma_x = \sigma_y = \sigma_z = -p, \quad \sigma_{xy} = 0, \quad \sigma_{yz} = \mu \frac{\partial w}{\partial y}, \quad \sigma_{zx} = \mu \frac{\partial w}{\partial x} \tag{1}$$

where $p$ denotes the pressure and $\mu$ the viscosity of the fluid.

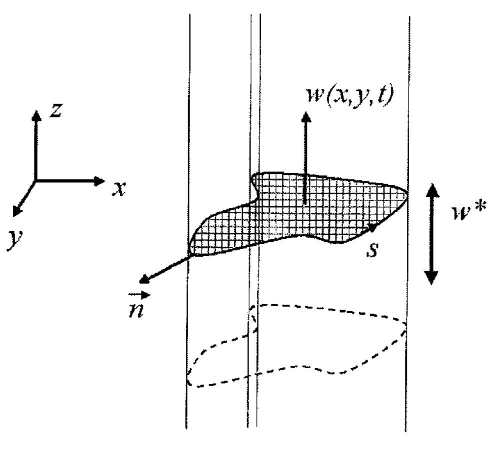


Figure 1. The principal model of the system.

The dynamic equilibrium equation in the direction of the $z$-axis can be written as

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right)-\frac{\partial p}{\partial z}+\rho g=\rho\frac{\partial w}{\partial t} \tag{2}$$

where $\rho$ is the density of the fluid, $g$ is the acceleration of gravity, and $\partial p/\partial z$ the gradient of the pressure in the direction of the $z$-axis, is assumed constant along the tube. It is also assumed that the fluid is non-Newtonian and that the viscosity is expressed as

$$\mu=\mu_1+\mu_2\sqrt{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2} \tag{3}$$

where $\mu_1$ and $\mu_2$ are constants. The non-Newtonian model given by Equation (3) presents an approximation for the pseudoplastic model often used for the description of the non-linear features of suspensions [11].

The wall boundary condition

$$-\mu\frac{\partial w}{\partial n}=\alpha(w-w^*) \tag{4}$$

takes into account the longitudinal vibration of the tube, where $n$ is the outward normal direction to the boundary of the cross-section of the flow inside the tube, $\alpha$ is the coefficient of slippage (sliding friction between the fluid and the surface of the tube) and $w^*$ is the velocity of the wall in the direction of the $z$-axis. It is assumed that the boundary performs harmonic oscillations in the direction of the $z$-axis, so that

$$w^*=a\sin\omega t \tag{5}$$

where $a$ and $\omega$ denote the amplitude and the frequency of oscillations. The amplitude of kinematic excitation of the boundary is $a/\omega$.

It can be noted that, for the pipe of circular cross-section, the problem is axi-symmetric but, bearing in mind that the results are applicable also for other tube shapes (to the tubes of other cross-sections), the 2D problem is solved. The cross-section of the flow is meshed using the finite element approximation. The resulting matrix differential equation:

$$[C]\{\dot{\delta}\}+[K]\{\delta\}=\{F\} \tag{6}$$

is obtained on the basis of the Galerkin method of weighted residuals [11, 12]. FEM matrices take the form:

$$\begin{aligned}
[C]&=\iint[N]^{\mathrm{T}}\rho[N]\,\mathrm{d}x\,\mathrm{d}y\\
[K]&=\iint[B]^{\mathrm{T}}\mu_1[B]\,\mathrm{d}x\,\mathrm{d}y+\oint[M]^{\mathrm{T}}\alpha[M]\,\mathrm{d}s\\
\{F\}&=\iint[N]^{\mathrm{T}}\left(\rho g-\frac{\partial p}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y-\iint[B]^{\mathrm{T}}\mu_2\sqrt{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2}[B]\{\delta\}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\oint[M]^{\mathrm{T}}\alpha w^*\,\mathrm{d}s
\end{aligned} \tag{7}$$

 

where $\{\delta\}$ is the vector of nodal velocities. The upper dot in Equation (6) denotes differentiation with respect to time; $s$ is the boundary line of the cross-section of the flow;

$$[N] = [N_1 \;\; N_2 \;\; \cdots \;\; N_n]$$

$$[B] = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \cdots & \dfrac{\partial N_n}{\partial x} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \cdots & \dfrac{\partial N_n}{\partial y} \end{bmatrix} \tag{8}$$

$$[M] = [M_1 \;\; M_2 \;\; \cdots \;\; M_m]$$

where $N_1, N_2 \ldots$ are the shape functions of the finite element in the cross-section of the flow, $M_1, M_2 \ldots$ are the shape functions of the finite element on the boundary of the cross-section of the flow.

The finite element equations could be integrated in time and transient flow processes can be analysed. Instead approximate methods of non-linear vibration theory are applied to this problem.

## HARMONIC BALANCE PROCEDURE

First the eigenpairs of the homogeneous system $[C]\{\dot{\delta}\} + [K]\{\delta\} = 0$ are determined and modal decomposition of Equation (6) is performed [12]:

$$[\Delta]^{\mathrm{T}}[C][\Delta]\{\dot{Z}\} + [\Delta]^{\mathrm{T}}[K][\Delta]\{Z\} = [\Delta]^{\mathrm{T}}\{F\} \tag{9}$$

where $[\Delta] = [\{\delta_1\} \;\; \{\delta_2\} \;\; \ldots]$ ; $\{\delta_i\}$ is the $i$th exponential decay eigenmode with corresponding eigenvalue $\lambda_i$. Then the vector of velocities is expressed as

$$\{\delta\} = [\Delta]\{Z\} = [\Delta]\begin{Bmatrix} z_1 \\ z_2 \\ \ldots \end{Bmatrix} \tag{10}$$

where $z_i$ denote modal coefficients for mode $i$.

The determination of the averaged surface of velocities is performed by solving the following four successive problems.

A. *Steady-state harmonic solution*. The sinusoidal variation of the wall velocity is assumed while the condition $-(\partial p/\partial z) + \rho g = 0$ is being satisfied, and the steady-state harmonic motion of the linear system is obtained by solving the modal equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}(z_i) + \lambda_i z_i = f_i \tag{11}$$

where $f_i$ is the modal loading for mode $i$. It is assumed that $\{F\} = \{F_s\} \sin \omega t$. Then, $\{F_s\} = \oint [M]^{\mathrm{T}} \alpha a \, \mathrm{d}s$ and $\begin{Bmatrix} f_1^s \\ f_2^s \\ \cdots \end{Bmatrix} = [\Delta]^{\mathrm{T}} \{F_s\}$. Thus $f_i = f_i^s \sin \omega t$, producing

$$z_i = z_i^s \sin \omega t + z_i^c \cos \omega t \tag{12}$$

The vectors of the sine and cosine components of the steady-state motion $\{\delta_s\}$ and $\{\delta_c\}$ are then expressed like:

$$\{\delta_s\} = [\Delta] \begin{Bmatrix} z_1^s \\ z_2^s \\ \cdots \end{Bmatrix}, \quad \{\delta_c\} = [\Delta] \begin{Bmatrix} z_1^c \\ z_2^c \\ \cdots \end{Bmatrix} \tag{13}$$

The following system of linear algebraic equations is solved for each eigenmode:

$$\begin{bmatrix} \lambda_i & -\omega \\ \omega & \lambda_i \end{bmatrix} \begin{Bmatrix} z_i^s \\ z_i^c \end{Bmatrix} = \begin{Bmatrix} f_i^s \\ 0 \end{Bmatrix} \tag{14}$$

B. *Static solution*. The static (constant in time) solution is obtained assuming that the term $-(\partial p/\partial z) + \rho g$ is constant in time and the wall velocity is equal to zero, that is:

$$\{F_0\} = \iint [N]^{\mathrm{T}} \left( \rho g - \frac{\partial p}{\partial z} \right) \mathrm{d}x \, \mathrm{d}y \tag{15}$$

This requires the solution of the system of linear algebraic equations:

$$[K]\{\delta_0\} = \{F_0\} \tag{16}$$

This solution again is sought using modal decomposition. Denoting $\begin{Bmatrix} f_1^0 \\ f_2^0 \\ \cdots \end{Bmatrix} = [\Delta]^{\mathrm{T}} \{F_0\}$ and $\{\delta_0\} = [\Delta] \begin{Bmatrix} z_1^0 \\ z_2^0 \\ \cdots \end{Bmatrix}$, for each eigenmode we have:

$$z_i^0 = \frac{f_i^0}{\lambda_i} \tag{17}$$

Up to this point the Newtonian model of the fluid is used as an approximation. The total velocities are:

$$\{\delta(t)\} = \{\delta_0\} + \{\delta_s\} \sin \omega t + \{\delta_c\} \cos \omega t \tag{18}$$

C. *Averaged transverse velocities*. The load vector is obtained in the process of assembly of the following loads:

(a) the load occurring from the constant term $-(\partial p/\partial z) + \rho g$;
(b) the non-linear term due to the static solution and the harmonic motion.

 

Those calculations are performed for a number of discrete moments equally spaced during the period. Time average of load vector takes the form:

$$\begin{aligned}\{F_1\} &= \overline{\iint [N]^{\mathrm{T}}\left(\rho g-\frac{\partial p}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y-\iint [B]^{\mathrm{T}}\mu_2\sqrt{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2}[B]\{\delta\}\,\mathrm{d}x\,\mathrm{d}y}\\ &= \iint [N]^{\mathrm{T}}\left(\rho g-\frac{\partial p}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y-\frac{1}{m}\sum_{i=1}^{m}\left(\iint [B]^{\mathrm{T}}\mu_2\sqrt{\left(\left.\frac{\partial w}{\partial x}\right|_{t=t_i}\right)^2+\left(\left.\frac{\partial w}{\partial y}\right|_{t=t_i}\right)^2}\right.\\ &\quad\times[B]\left\{\{\delta_0\}+\{\delta_s\}\sin\left(\frac{2\pi}{m}(i-1)\right)\right.\\ &\quad\left.\left.+\{\delta_c\}\cos\left(\frac{2\pi}{m}(i-1)\right)\right\}\mathrm{d}x\,\mathrm{d}y\right)\end{aligned}\tag{19}$$

where the upper dash denotes time average; $\omega t=(2\pi/m)(i-1)$;

$$\left\{\begin{matrix}\left.\dfrac{\partial w}{\partial x}\right|_{t=t_i}\\ \left.\dfrac{\partial w}{\partial y}\right|_{t=t_i}\end{matrix}\right\}=[B]\left\{\{\delta_0\}+\{\delta_s\}\sin\left(\frac{2\pi}{m}(i-1)\right)+\{\delta_c\}\cos\left(\frac{2\pi}{m}(i-1)\right)\right\}\tag{20}$$

$m$ is the number of discrete time moments in a period.

Finally, the non-linear problem is solved obtaining the averaged surface of velocities using previously described technique of modal decomposition for the solution of the system of linear algebraic equations:

$$[K]\{\delta_1\}=\{F_1\}\tag{21}$$

D. *Mass flow rate*. Finally, the average mass flow rate is found by integrating over the cross-sectional area:

$$Q=\iint \rho w(x,y)\,\mathrm{d}x\,\mathrm{d}y\tag{22}$$

where $w(x,y)$ are the averaged transverse velocities which are calculated from $\{\delta_1\}$ by using the shape functions of the appropriate finite elements. It is assumed that the averaging interval is much longer than the period of oscillations.

## NUMERICAL RESULTS

Cross-section of the tube is assumed to be a circle and one-fourth of it is analysed (Figure 2). The characteristics of the non-Newtonian fluid represent a liquid-type suspension ($\mu_1=0.004$ g/mm s $=4$ cP, $\mu_2=-0.0001$ g/mm, $\alpha=0.04$ g/mm$^2$ s, $\rho=0.001$ g/mm$^3$, $-(\partial p/\partial z)+\rho g=0.003$ g/mm$^2$ s$^2$). Radius of the tube is $R=10$ mm; number of moments in the period used for

 

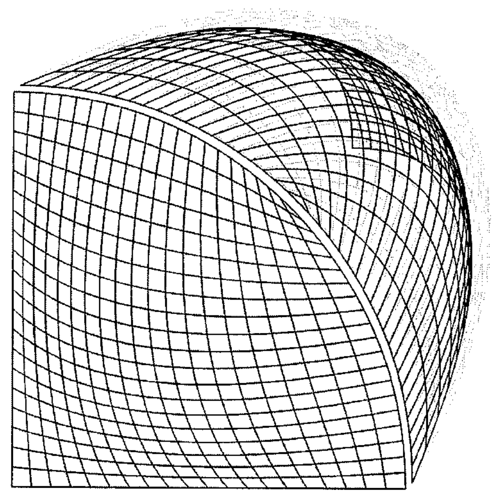

Figure 2. The finite element mesh and the oblique projection of cross-flow velocities—non-excited (black) and excited (grey) at $a/\omega = 4\,\text{mm}$; $\omega = 2\,\text{rad/s}$; projection angle 45°; projected depth 0.6 of width.

averaging is $m = 16$. The procedure of harmonic balance is validated comparing the results with direct time integration results.

Longitudinal tube vibration increases the flow velocities in the cross-section at the same pressure gradient. Such effect takes place due to the non-linearity of the fluid. Improvement of Newtonian fluid flow would be impossible by harmonic excitation of the boundary. Numerical experiments show that the mass flow rate grows with the excitation amplitude and frequency. Such analysis enables the selection of optimal frequencies and feasible levels of amplitudes of excitation.

Numerical validation of harmonic balance procedure was performed by comparison of the average mass flow rate with results of direct time integration (Figure 2).

Averaged surfaces of cross-sectional velocities with and without external excitation are presented in Figure 3. It can be seen how external excitation of the tube increases the flow velocities in the cross-section (Figure 3).

The relationship between the mass flow rate and the amplitudes of excitation at different frequencies of excitation is presented in Figure 4.

The accuracy and the cost of the standard time integration approaches depend mainly on the quality of meshing, the selection of the time step and the choice of the criterion for distinguishing steady-state flows from transient flows. The choice of the optimal values of the mentioned factors is usually non-trivial and may require heavy computational efforts. The main advantage of the proposed harmonic balance procedure is that the steady-state regimes can be determined directly if only the excitation is harmonic. This could be especially advantageous in the design stage of different fluid dosing elements when mass flow rates must be optimized in broad parameter domains.

It can be noted that the calculation of the instantaneous mass flow rate must be performed for each time step when performing time integration—it is used for the determination of the average mass flow rate over the last calculated period of steady-state motion. Assuming that

 

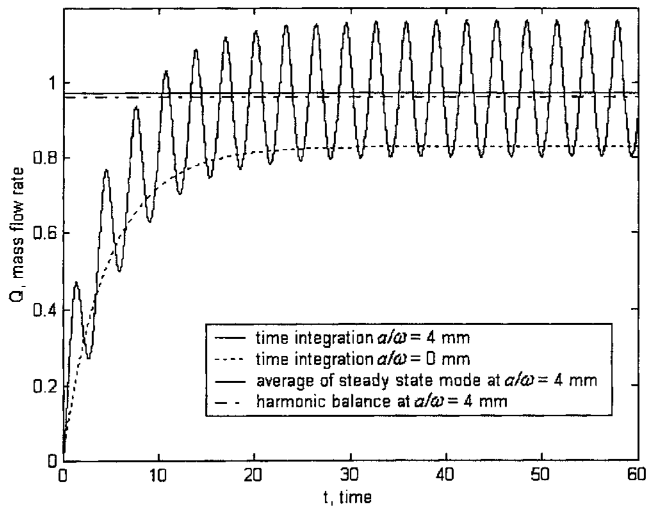


Figure 3. Numerical validation of the harmonic balance procedure.

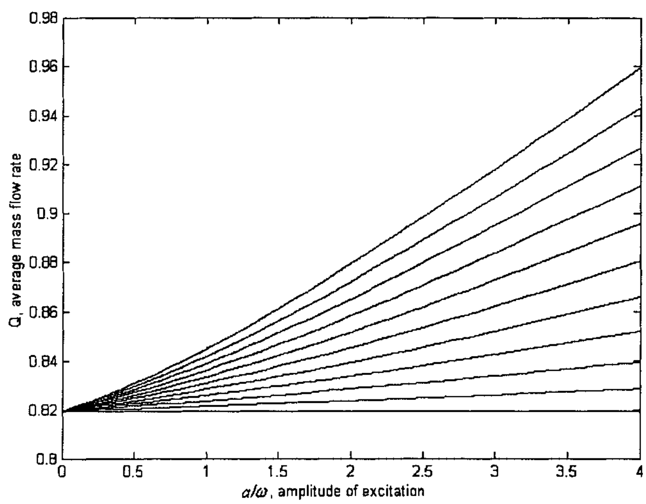


Figure 4. Relationship between the mass flow rate $Q(g/s)$ and the amplitude of excitation $a/\omega$ (mm) at various frequencies of excitation $\omega_i = (i-1) \cdot 0{,}2;\ i = 1, \ldots, 21$.

the cost of numerical integration over one time step is $W$; the calculation of the mass flow rate is performed for each time step at a cost $w$; $m$ time steps in a period of excitation are used; 10 periods are required to achieve steady-state motion, the cost of numerical integration can be approximated as $10m(W + w)$.

 

The costs of the stages of the harmonic balance procedure for the same problem can be approximated:

A. Steady-state harmonic solution—cost is $W$ (comparable with one time step for the same meshing).
B. Static solution—cost is $W/2$ (about 50% of the cost of harmonic solution).
C. Determination of the averaged transverse velocities—cost is $mW/2$ ($m$ is the number of discrete time moments in a period of excitation);
D. Determination of the average mass flow rate—cost is $w$.

Thus the total cost of the proposed harmonic balance procedure is $((3+m)/2)\cdot W + w$. For realistic meshings $w$ can be approximated as $0.1\,W$. So the cost of the proposed harmonic balance procedure is about twenty times lower than that of numerical time integration. Numerical experiments indicate that the proposed harmonic balance procedure produces results which in some cases can differ by about 10% from the 'exact' ones obtained from numerical integration.

The problem for which the harmonic balance procedure was developed is typical for the transportation of fluid and therefore should be regarded as a demonstrational problem for developing the algorithm. The potential of the method lies in its possible applications to the three dimensional non-linear problems of computational fluid dynamics.

## CONCLUSIONS

Numerical model describing flow of non-Newtonian fluid in a tube performing longitudinal vibrations is developed. It must be noted that this model incorporates non-linearity and represents the dynamic behaviour of the liquid suspension. The inertia of the fluid is taken into account and FEM damping matrix is obtained. The velocity of the longitudinal motion of the walls is taken into account through the boundary condition and thus is directly incorporated into the finite element formulation. Harmonic balance method for determination of averaged transverse velocities of non-Newtonian fluid is proposed. The results of the analysis show that the longitudinal tube vibrations can be effectively applied for fluid flow control and incorporated into the design of vibration spraying and dosing devices of various substances.

## REFERENCES

1. Hui L, Tomita Y. A numerical simulation of swirling flow pneumatic conveying in a horizontal pipeline. *Particulate Science and Technology: An International Journal* 2000; **18**(4):275–292.
2. Hauser G, Sommer K. Plug flow type pneumatic conveying of abrasive products in food industry. *Journal of Powder and Bulk Solids Technology* 1998; **12**:511–519.
3. Bubulis A, Ragulskis L. Dry substances transportation and dosing using vibrations. *41 Internationales Wissenchaftliches Kolloquium*, Ilmenau, 1996; 492–496.
4. Ragulskis M, Palevicius A. Theoretical and experimental studies of tubular valve dynamics. *International Journal of Acoustics and Vibration* 2002; **7**(1):53–57.
5. Morgan K, Peraire J. Unstructured grid finite element methods for fluid dynamics. *Reports on Progress in Physics* 1998; **61**(6):569–638.
6. Thomasset F. *Implementation of Finite Element Methods for Navier–Stokes Equations*. Springer: Germany, 1981; 159.
7. Blawzdziewicz J, Feuillebois F. Calculation of an effective slip in a settling suspension at a vertical wall. *International Journal of Fluid Mechanics Research* 1995; **22**(1):164–169.
8. Baker AJ. *Finite Element Computational Fluid Mechanics*. Hemisphere PC: U.S.A., 1983; 510.

 

9. Huang Y, Hsu C. Dynamic behaviour of tubes subjected to internal and external cross flows. *Journal of Shock and Vibration* 1997; **4**(2):77–91.
10. Jordan DW, Smith P. *Nonlinear Ordinary Differential Equations*. Oxford University Press: U.K., 1999; 550.
11. Rao SS. *The Finite Element Method in Engineering*. Pergamon Press: U.K., 1982; 625.
12. Zienkiewicz OC, Morgan K. *Finite Elements and Approximation*. Mir: Moscow, 1986; 320.
13. Hearn D, Baker MP. *Computer Graphics*. Prentice-Hall International: U.S.A., 1986; 351.

 