THE LOGISTIC MAP OF MATRICES

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Abstract. The standard iterative logistic map is extended by replacing the scalar variable by a square matrix of variables. Dynamical properties of such an iterative map are explored in detail when the order of matrices is 2. It is shown that the evolution of the logistic map depends not only on the control parameter but also on the eigenvalues of the matrix of initial conditions. Several computational examples are used to demonstrate the convergence to periodic attractors and the sensitivity of chaotic processes to initials conditions.

1. **Introduction.** The logistic map is a paradigmatic model often used to demonstrate the onset of chaos and to illustrate how complex behavior can arise from very simple non-linear dynamical equations [1, 2]:

$$x^{(n+1)} = ax^{(n)} \left(1 - x^{(n)} \right);$$ (1)

where n is the iteration number; $n = 0, 1, 2, \ldots$; $a \in R$ is the parameter of the logistic map and $x^{(0)}$ is the initial condition (the initial population at year 0). The logistic map is thoroughly explored and is used to model [3, 4, 5, 6], encrypt [7, 8], predict [9, 10] different physical systems and processes. A number of extensions of the logistic map have been proposed. The logistic map in two dimensions is introduced in [11]; the bi-parameter logistic map is used to model a car following model in [12]; a complex logistic map is used to generate fractals in [13]; the concept of the random logistic map is introduced in [14]; the interplay between noise and chaos in the stochastic logistic map is investigated in [15]; the coupled logistic map is used to study the effects of spatial heterogeneity on population dynamics in [16]; a two-dimensional logistic coupled map lattice is exploited to describe the Turing

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instability in [17]; averaged logistic maps are used to construct carrying surfaces of return maps in [18]; the compound logistic map is investigated in [19].
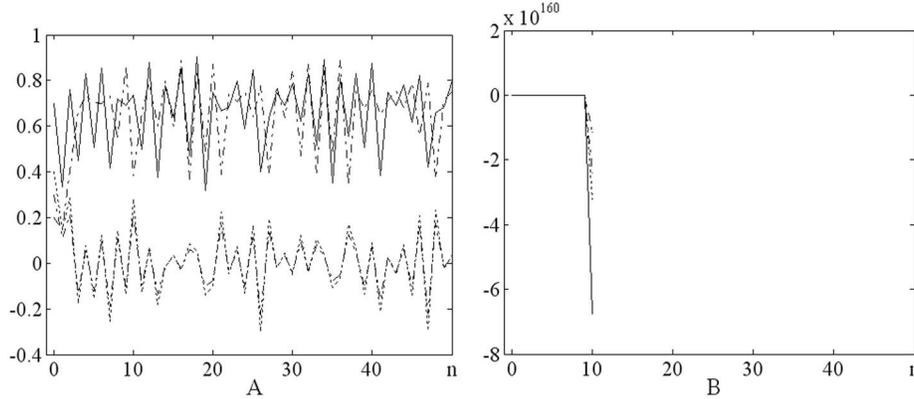

FIGURE 1. The controversy of the logistic map of matrices: $X^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{smallmatrix}\right]$ results into 4 stationary processes (A) while $X^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.9 \end{smallmatrix}\right]$ yields a violent divergence of iterative processes (B); $a = 3.7$; dot-dashed lines stand for $x_{11}^{(n)}$; dashed-dotted lines stand for $x_{12}^{(n)}$; dashed lines stand for $x_{21}^{(n)}$ and thin solid lines stand for $x_{22}^{(n)}$.

The object of this paper is to investigate the extension of the logistic map when the discrete scalar variable $x^{(n)}$ is replaced by a square matrix of order 2; the n-th iterate of that matrix is denoted as $X^{(n)}$. Let the matrix of initial conditions reads:
$X^{(0)} = \begin{bmatrix} x_{11}^{(0)} & x_{12}^{(0)} \\ x_{21}^{(0)} & x_{22}^{(0)} \end{bmatrix}$; $x_{kl}^{(0)} \in R$; $k, l = 1, 2$. Then the iterated map

$$X^{(n+1)} = aX^{(n)}\left(I - X^{(n)}\right) := \begin{bmatrix} x_{11}^{(n+1)} & x_{12}^{(n+1)} \\ x_{21}^{(n+1)} & x_{22}^{(n+1)} \end{bmatrix}, \tag{2}$$

represents a logistic map of square matrices of order 2. Eq. (2) produces four scalar time series $\left\{x_{kl}^{(j)}\right\}_{j=0}^{+\infty}$; $k, l = 1, 2$. Explicitly,

$$\begin{cases} x_{11}^{(n+1)} = a\left(x_{11}^{(n)}\left(1 - x_{11}^{(n)}\right) - x_{12}^{(n)}x_{21}^{(n)}\right); \\ x_{12}^{(n+1)} = ax_{12}^{(n)}\left(1 - x_{11}^{(n)} - x_{22}^{(n)}\right); \\ x_{21}^{(n+1)} = ax_{21}^{(n)}\left(1 - x_{11}^{(n)} - x_{22}^{(n)}\right); \\ x_{22}^{(n+1)} = a\left(x_{22}^{(n)}\left(1 - x_{22}^{(n)}\right) - x_{12}^{(n)}x_{21}^{(n)}\right); \end{cases} \tag{3}$$

$n = 0, 1, 2, \ldots$; and $x_{11}^{(0)}, x_{12}^{(0)}, x_{21}^{(0)}, x_{22}^{(0)}$ are four scalar initial conditions. Though such an extension of the classical logistic map seems to be trivial, the apparent simplicity of the dynamical properties of such an iterative map is misguiding. As an example let us select two different sets of initial conditions and follow the evolution of four time series (at fixed parameter value $a = 3.7$). Initial conditions $X^{(0)} = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$ yield 4 fluctuating processes (Fig. 1A - note that some values of

iterated time series are lower than 0). But initial conditions $X^{(0)} = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.9 \end{bmatrix}$ yield a violent divergence of iterative processes; numerical overflow is reached after 10 iterations only (Fig. 1B). The primary object of this paper is to explain such dynamic behavior of the logistic map of matrices when the scalar discrete variable is replaced by a square matrix or order 2.

2. **Auxiliary Results.** Several properties of square matrices of order 2 will be discussed in this section. These properties are essential before continuing with the logistic map of matrices.

2.1. **Algebraic representation of matrices.** Let us consider a square matrix of order 2:

$$X := \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \tag{4}$$

$x_{11}, \ldots, x_{22} \in C$ and its eigenvalues $\lambda_1, \lambda_2 \in C$:

$$\lambda_{1,2} = \frac{1}{2} \left(\operatorname{Tr} X \pm \sqrt{\operatorname{dsk} X} \right), \tag{5}$$

where $\operatorname{Tr} X := x_{11} + x_{22}$; $\operatorname{dsk} X := (x_{11} - x_{22})^2 + 4x_{12}x_{21}$.

Corollary 1. *Let eigenvalues of the matrix X are not equal: $\lambda_1 \neq \lambda_2$. Then it is possible to construct two matrices D_k:*

$$D_k := \frac{1}{\lambda_k - \lambda_l} \left(X - \lambda_l I \right); \quad k, l = 1, 2; \quad k \neq l; \tag{6}$$

where I is the identity matrix; $I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Matrices D_k satisfy following equalities:

(i) $\det D_k = 0$;
(ii) $D_1 + D_2 = I$;
(iii) $D_k \cdot D_l = \delta_{kl} D_k$; $k, l = 1, 2$ where $\delta_{kl} := \begin{cases} 1, & k = l; \\ 0, & k \neq l. \end{cases}$

Proof. The equality (i) follows from Eq. (6) because $\det X - \lambda_i I = 0$. The equality (ii) holds because $D_1 + D_2 = \frac{1}{\lambda_1 - \lambda_2} (A - \lambda_2 I) + \frac{1}{\lambda_2 - \lambda_1} (A - \lambda_1 I) = I$.

The proof of the equality (iii) is straightforward. Let $k = l = 1$. Then, Cayley-Hamilton theorem [20] and Eq. (6) yield:

$$D_1^2 = \frac{1}{(\lambda_1 - \lambda_2)^2} (X - \lambda_2 I)^2 = \frac{1}{(\lambda_1 - \lambda_2)^2} \left(X^2 - 2\lambda_2 X + \lambda_2^2 I \right)$$

$$= \frac{1}{(\lambda_1 - \lambda_2)^2} \left(\left(X^2 - (\lambda_1 + \lambda_2) X + \lambda_1 \lambda_2 I \right) - \lambda_2 X + \lambda_1 X + \lambda_2^2 I - \lambda_1 \lambda_2 I \right)$$

$$= \frac{1}{(\lambda_1 - \lambda_2)^2} \left((\lambda_1 - \lambda_2) X - (\lambda_1 - \lambda_2) \lambda_2 I \right) = \frac{1}{\lambda_1 - \lambda_2} (X - \lambda_2 I) = D_1.$$

Analogously it can be proven that $D_2^2 = D_2$. Let now $k = 1$ and $l = 2$. Then,

$$D_1 \cdot D_2 = \frac{1}{\lambda_1 - \lambda_2} \left(X - \lambda_2 I\right) \cdot \frac{1}{\lambda_2 - \lambda_1} \left(X - \lambda_1 I\right)$$
$$= -\frac{1}{\left(\lambda_1 - \lambda_2\right)^2} \left(X^2 - \left(\lambda_1 + \lambda_2\right) X + \lambda_1 \cdot \lambda_2 I\right) = \Theta;$$

where $\Theta := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Analogously it can be shown that $D_2 \cdot D_1 = \Theta$. $\qquad \square$

Corollary 2. *Let eigenvalues of the matrix X coincide: $\lambda_1 = \lambda_2 = \lambda_0$. Then the matrix N defined as:*

$$N := X - \lambda_0 I \qquad (7)$$

satisfies following relationships:
  (i) $N^2 = \Theta$;
  (ii) $\det N = 0$.

Proof. The equality (i) holds because Cayley – Hamilton theorem yields: $N^2 = (X - \lambda_0 I)^2 = X^2 - 2\lambda_0 X + \lambda_0^2 I = \Theta$. The equality (ii) follows from the property of a determinant of the product of matrices. $\qquad \square$

Corollary 3. *If $\lambda_1 \neq \lambda_2$ (dsk $X \neq 0$) then the matrix X can be expressed as:*

$$X = \lambda_1 D_1 + \lambda_2 D_2. \qquad (8)$$

If $\lambda_1 = \lambda_2 = \lambda_0$ (dsk $X = 0$) then the matrix X can be expressed as:

$$X = \lambda_0 I + N. \qquad (9)$$

Proof. Eq. (9) holds because:

$$\lambda_1 D_1 + \lambda_2 D_2 = \frac{\lambda_1}{\lambda_1 - \lambda_2} \left(X - \lambda_2 I\right) + \frac{\lambda_2}{\lambda_2 - \lambda_1} \left(X - \lambda_1 I\right)$$
$$= \left(\frac{\lambda_1}{\lambda_1 - \lambda_2} + \frac{\lambda_2}{\lambda_2 - \lambda_1}\right) X = X \qquad (10)$$

The validity of Eq. (9) follows from Eq. (7). $\qquad \square$

Let D_1 is an idempotent $(D_1^2 = D_1)$. Then $D_2 := I - D_1$ is also an idempotent because $D_2^2 = (I - D_1)^2 = I^2 - 2D_1 + D_1^2 = I - D_1$. Analogously, let N is a nilpotent. Then cN is a nilpotent also; $c \in C$. Idempotents D and $I - D$ are conjugate idempotents. Nilpotents N and cN; $c \in C$ are similar nilpotents. (N and Θ are similar nilpotents).

Definition 2.1. The matrix X is an idempotent matrix if it can be expressed in the form of Eq. (8) where $\lambda_1; \lambda_2$ are eigenvalues of X and $D_1; D_2$ are conjugate idempotents $(D_1 + D_2 = I)$. The matrix X is a nilpotent matrix if its eigenvalues are equal $\lambda_1 = \lambda_2 = \lambda_0$ and Eq. (9) holds.

Corollary 4. *Matrices $D_1 = 1 \cdot D_1 + 0 \cdot D_2$ and $D_2 = 0 \cdot D_1 + 1 \cdot D_2$ are idempotent matrices, and matrix $N = 0 \cdot I + N$ is a nilpotent matrix.*

Corollary 5. *The matrix X is an idempotent matrix if its eigenvalues are not equal; the matrix X is a nilpotent matrix if its eigenvalues are equal. Let us notice that a scalar matrix $X = \lambda_0 I$ can be expressed in the form $\lambda_0 I = \lambda_0 D_1 + \lambda_0 D_2$ where $D_1; D_2$ is a pair of conjugate idempotents. Thus, $\lambda_0 I$ can be interpreted as a nilpotent $(\lambda_0 I = \lambda_0 I + \Theta)$ or as an idempotent matrix.*

Example 1. This will illustrate that a scalar matrix can interpreted as a nilpotent or an idempotent matrix. Let us assume that $\lambda_0 = 0.2$. Then,

$$
\begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{5} \left(\begin{bmatrix} \frac{3}{2} & \frac{5}{2} \\ -\frac{3}{10} & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} \\ \frac{3}{10} & \frac{3}{2} \end{bmatrix} \right)
$$

It can be noted that matrices $\begin{bmatrix} \frac{3}{2} & \frac{5}{2} \\ -\frac{3}{10} & -\frac{1}{2} \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} \\ \frac{3}{10} & \frac{3}{2} \end{bmatrix}$ are conjugate idempotents.

Corollary 6. *Let idempotents of idempotent matrices X_1' and X_1'' are the same. Then idempotents of $X_1' \cdot X_1''$ and $X_1' + X_1''$ are also the same. Analogously, let nilpotents of nilpotent matrices X_2' and X_2'' are similar. Then the nilpotent of $X_2' \cdot X_2''$ and the nilpotent of $X_2' + X_2''$ is similar to the nilpotent of X_2' and the nilpotent of X_2''.*

Proof. Let $X_1' = \lambda_1' D_1 + \lambda_2' D_2$ and $X_1'' = \lambda_1'' D_1 + \lambda_2'' D_2$. Then

$$
\begin{aligned}
X_1' + X_1'' &= (\lambda_1' D_1 + \lambda_2' D_2) + (\lambda_1'' D_1 + \lambda_2'' D_2) \\
&= (\lambda_1' + \lambda_1'')D_1 + (\lambda_2' + \lambda_2'')D_2
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
X_1' \cdot X_1'' &= (\lambda_1' D_1 + \lambda_2' D_2)(\lambda_1'' D_1 + \lambda_2'' D_2) \\
&= \lambda_1'\lambda_1'' D_1^2 + \lambda_2'\lambda_1'' D_2 D_1 + \lambda_1'\lambda_2'' D_1 D_2 + \lambda_2'\lambda_2'' D_2^2 \\
&= \lambda_1'\lambda_1'' D_1 + \lambda_2'\lambda_2'' D_2
\end{aligned}
\tag{12}
$$

Analogously, let $X_2' = \lambda_0' I + c_1 N$ and $X_2'' = \lambda_0'' I + c_2 N$

$$
\begin{aligned}
X_2' + X_2'' &= (\lambda_0' I + c_1 N) + (\lambda_0'' I + c_2 N) \\
&= (\lambda_0' + \lambda_0'')I + (c_1 + c_2)N.
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
X_2' \cdot X_2'' &= (\lambda_0' I + c_1 N)(\lambda_0'' I + c_2 N) \\
&= \lambda_0'\lambda_0'' I \cdot I + c_1\lambda_0'' N + \lambda_0' c_2 N + c_1 c_2 N^2 \\
&= \lambda_0'\lambda_0'' I + (c_1\lambda_0'' + c_2\lambda_0')N.
\end{aligned}
\tag{14}
$$

$\square$

Corollary 7. *Let X_1 is an idempotent matrix and X_2 is a nilpotent matrix. Then powers X_1^n and X_2^n; $n = 0, 1, 2, \dots$ read:*

$$
X_1^n = \lambda_1^n D_1 + \lambda_2^n D_2;
\tag{15}
$$

$$
X_2^n = \lambda_0^n I + n\lambda_0^{n-1} N;
\tag{16}
$$

where D_1 and D_2 are idempotents of X_1 and N is the nilpotent of X_2.

Proof.

$$X_1^2 = (\lambda_1 D_1 + \lambda_2 D_2)^2 = \lambda_1^2 D_1^2 + \lambda_1 \lambda_2 D_1 \cdot D_2 + \lambda_2 \lambda_1 D_2 \cdot D_1 + \lambda_2^2 D_2^2$$
$$= \lambda_1^2 D_1 + \lambda_2^2 D_2; \quad \dots$$
$$X_2^2 = (\lambda_0 I + N)^2 = \lambda_0^2 I + 2\lambda_0 N + N^2 = \lambda_0^2 I + 2\lambda_0 N; \quad \dots .$$

$\square$

Eq. (15) and Eq. (16) hold for $n \in Z$; the proof is analogous to the proof of Corollary 7.

Complex number theory can be exploited to generalize Eq. (15) and Eq. (16) for $n \in R$, but this is out of scope of interests in this paper.

Corollary 8. *Let two conjugate idempotents D_1 and D_2 and two constants $\lambda_1, \lambda_2 \in C$ are given. Then λ_1, λ_2 are eigenvalues and D_1, D_2 are conjugate idempotents of a matrix $X := \lambda_1 D_1 + \lambda_2 D_2$.*

Proof. Let $\lambda_1 \neq \lambda_2$. Then the matrix characteristic equation

$$\det\left[(\lambda_1 D_1 + \lambda_2 D_2) - \lambda I\right] = \det\left[(\lambda_1 - \lambda) D_1 + (\lambda_2 - \lambda) D_2\right] = 0$$

yields two solutions $\lambda = \lambda_1$ and $\lambda = \lambda_2$. Therefore λ_1 and λ_2 are eigenvalues of X. Also, $\frac{1}{\lambda_1 - \lambda_2} (\lambda_1 D_1 + \lambda_2 D_2 - \lambda_2 I) = \frac{1}{\lambda_1 - \lambda_2} (\lambda_1 - \lambda_2) D_1 = D_1$. Therefore D_1 and D_2 are idempotents of X. Finally, if $\lambda_1 = \lambda_2 = \lambda_0$, then X is a scalar matrix and it has one recurrent eigenvalue λ_0. $\square$

Corollary 9. *Let a nilpotent N and a constant $\lambda_0 \in C$ are given. Then a matrix $X := \lambda_0 I + N$ has a single recurrent eigenvalue λ_0 and its nilpotent is N.*

The proof is analogous to the proof of Corollary 8.

2.2. **Parametric expressions of idempotents and nilpotents.** It can be noted that eigenvalues λ_1 ir λ_2 of a second order idempotent matrix X do satisfy the following relationships:

$$\lambda_1 - \lambda_2 = \sqrt{\mathrm{dsk}\ X}; \tag{17}$$

$$x_{11} - \lambda_2 = x_{11} - \frac{1}{2}\left(x_{11} + x_{22} - \sqrt{\mathrm{dsk}\ X}\right) = \frac{1}{2}\left(\sqrt{\mathrm{dsk}\ X} + (x_{11} - x_{22})\right); \tag{18}$$

and analogously,

$$x_{22} - \lambda_1 = -\frac{1}{2}\left(\sqrt{\mathrm{dsk}\ X} - (x_{22} - x_{11})\right). \tag{19}$$

Eq. (6) yields:

$$D_1 = \frac{1}{2\sqrt{\mathrm{dsk}\ X}} \begin{bmatrix} \sqrt{\mathrm{dsk}\ X} + (x_{11} - x_{22}) & 2x_{12} \\ 2x_{21} & \sqrt{\mathrm{dsk}\ X} + (x_{22} - x_{11}) \end{bmatrix}; \tag{20}$$

and

$$D_2 = \frac{1}{2\sqrt{\mathrm{dsk}\ X}} \begin{bmatrix} \sqrt{\mathrm{dsk}\ X} + (x_{22} - x_{11}) & -2x_{12} \\ -2x_{21} & \sqrt{\mathrm{dsk}\ X} + (x_{11} - x_{22}) \end{bmatrix} \tag{21}$$

when dsk $X \neq 0$. The expression of the nilpotent follows analogously:

$$N = \frac{1}{2} \begin{bmatrix} x_{11} - x_{22} & 2x_{12} \\ 2x_{21} & x_{22} - x_{11} \end{bmatrix} \tag{22}$$

when dsk $X = 0$ and $\lambda_0 = \frac{x_{11}+x_{22}}{2}$.

Let us assume that X is an idempotent matrix. Then the introduction of new parameters $\alpha := \frac{x_{11}-x_{22}}{\sqrt{\text{dsk } X}}$; $\beta := \frac{2x_{12}}{\sqrt{\text{dsk } X}}$ and Eq. (5, 6) yield the parametric expression of idempotents of X:

$$D_1 = \frac{1}{2} \begin{bmatrix} 1+\alpha & \beta \\ \frac{1-\alpha^2}{\beta} & 1-\alpha \end{bmatrix}; D_2 = \frac{1}{2} \begin{bmatrix} 1-\alpha & -\beta \\ -\frac{1-\alpha^2}{\beta} & 1+\alpha \end{bmatrix}; \tag{23}$$

because the definition of dsk X yields:

$$\left(\frac{(x_{11}-x_{22})}{\sqrt{\text{dsk } X}} \right)^2 + \frac{2x_{12}}{\sqrt{\text{dsk } X}} \cdot \frac{2x_{21}}{\sqrt{\text{dsk } X}} = 1$$

and

$$\frac{2x_{21}}{\sqrt{\text{dsk } X}} = \frac{1-\alpha^2}{\beta}.$$

It can be noted that $\alpha = 2$ and $\beta = 5$ in Example 1.

Analogously, when X is a nilpotent matrix, notations $\hat{\alpha} = x_{11} - x_{22}$; $\hat{\beta} := 2x_{12}$; Eq. (7) and the equality dsk $X = 0$ yield:

$$N = \frac{1}{2} \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\frac{\hat{\alpha}^2}{\hat{\beta}} & -\hat{\alpha} \end{bmatrix}. \tag{24}$$

Thus, parametric expressions of idempotents D_1, D_2 and the nilpotent N exist for all values of parameters $\alpha, \hat{\alpha}, \beta, \hat{\beta} \in C$ except $\beta, \hat{\beta} = 0$. It can be noted that other notions of parameters would lead to different parametric expressions of idempotents and the nilpotent.

It can be noted that following conditional parametric limits exist:

$$\lim_{\substack{\alpha \to 1 \\ \beta \to 0 \\ \left(\frac{1-\alpha^2}{\beta} \to 0 \right)}} D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \qquad \lim_{\substack{\alpha \to 1 \\ \beta \to 0 \\ \left(\frac{1-\alpha^2}{\beta} \to 0 \right)}} D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};$$

$$\lim_{\substack{\hat{\alpha} \to 0 \\ \hat{\beta} \to 0 \\ \left(\frac{-\hat{\alpha}^2}{2\hat{\beta}} \to b \right)}} N = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}; \qquad \lim_{\substack{\hat{\alpha} \to 0 \\ \frac{\hat{\beta}}{2} \to b}} N = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix};$$

where $b \in C$.

These limit transformations help to identify all possible second order idempotents and nilpotents.

3. **The dynamics of the logistic map of matrices.** We will highlight main features of the dynamics of the logistic map of square matrices of order 2 in this section.

3.1. The matrix of initial conditions, its eigenvalues and iterative processes.

Theorem 3.1. *Four iterated sequences* $\left\{ x_{kl}^{(n)} \right\}_{n=0}^{+\infty}$; $k, l = 1, 2$ *generated by the logistic map of matrices defined by Eq.* (2) *will stay bounded for all* $n = 0, 1, 2, \ldots$ *if following statements hold:*

(i) $0 \leq a \leq 4$;
(ii) *the matrix of initial conditions* $X^{(0)}$ *is an idempotent matrix;*
(iii) *eigenvalues* $\lambda_1^{(0)}$ *and* $\lambda_2^{(0)}$ *of* $X^{(0)}$ *are bounded in the interval* $[0; 1]$.

Proof. Since $X^{(0)}$ is an idempotent matrix:

$$X^{(0)} = \begin{bmatrix} x_{11}^{(0)} & x_{12}^{(0)} \\ x_{21}^{(0)} & x_{22}^{(0)} \end{bmatrix} = \lambda_1^{(0)} \frac{1}{2} \begin{bmatrix} 1+\alpha & \beta \\ \frac{1-\alpha^2}{\beta} & 1-\alpha \end{bmatrix} + \lambda_2^{(0)} \frac{1}{2} \begin{bmatrix} 1-\alpha & -\beta \\ -\frac{1-\alpha^2}{\beta} & 1+\alpha \end{bmatrix}. \quad (25)$$

Straightforward computations (Eq. (2)) yield: $X^{(1)} = \lambda_1^{(1)} D_1 + \lambda_2^{(1)} D_2$; where D_1 and D_2 are defined in Eq. (23). Then, $X^{(n+1)}$ takes the following form:

$$\begin{aligned}
X^{(n+1)} &= a \left(\lambda_1^{(n)} D_1 + \lambda_2^{(n)} D_2 \right) \left(I - \lambda_1^{(n)} D_1 - \lambda_2^{(n)} D_2 \right) \\
&= a \left(\lambda_1^{(n)} D_1 + \lambda_2^{(n)} D_2 - \left(\lambda_1^{(n)} \right)^2 (D_1)^2 - \lambda_1^{(n)} \lambda_2^{(n)} D_2 D_1 \right. \\
&\quad \left. - \lambda_1^{(n)} \lambda_2^{(n)} D_1 D_2 - \left(\lambda_2^{(n)} \right)^2 (D_2)^2 \right) \\
&= a \left(\lambda_1^{(n)} D_1 + \lambda_2^{(n)} D_2 - \left(\lambda_1^{(n)} \right)^2 D_1 - \left(\lambda_2^{(n)} \right)^2 D_2 \right) \\
&= a \left(\lambda_1^{(n)} - \left(\lambda_1^{(n)} \right)^2 \right) D_1 + a \left(\lambda_2^{(n)} - \left(\lambda_2^{(n)} \right)^2 \right) D_2 \\
&= \lambda_1^{(n+1)} D_1 + \lambda_2^{(n+1)} D_2.
\end{aligned} \quad (26)$$

It is clear that the structure of $X^{(n)}$ is identical to the structure of $X^{(0)}$ (the introduction Eq. (25) into Eq. (2) produces an iterative sequence of matrices):

$$\begin{aligned}
X^{(n)} &= \begin{bmatrix} x_{11}^{(n)} & x_{12}^{(n)} \\ x_{21}^{(n)} & x_{22}^{(n)} \end{bmatrix} \\
&= \lambda_1^{(n)} \frac{1}{2} \begin{bmatrix} 1+\alpha & \beta \\ \frac{1-\alpha^2}{\beta} & 1-\alpha \end{bmatrix} + \lambda_2^{(n)} \frac{1}{2} \begin{bmatrix} 1-\alpha & -\beta \\ -\frac{1-\alpha^2}{\beta} & 1+\alpha \end{bmatrix}.
\end{aligned} \quad (27)$$

Eq. (27) and Eq. (2) yield:

$$\lambda_1^{(n+1)} \frac{1}{2} \begin{bmatrix} 1+\alpha & \beta \\ \frac{1-\alpha^2}{\beta} & 1-\alpha \end{bmatrix} + \lambda_2^{(n+1)} \frac{1}{2} \begin{bmatrix} 1-\alpha & -\beta \\ -\frac{1-\alpha^2}{\beta} & 1+\alpha \end{bmatrix}$$

$$= a \left(\left(\lambda_1^{(n)} - \left(\lambda_1^{(n)} \right)^2 \right) \frac{1}{2} \begin{bmatrix} 1+\alpha & \beta \\ \frac{1-\alpha^2}{\beta} & 1-\alpha \end{bmatrix} \right. \tag{28}$$

$$\left. + \left(\lambda_2^{(n)} - \left(\lambda_2^{(n)} \right)^2 \right) \frac{1}{2} \begin{bmatrix} 1-\alpha & -\beta \\ -\frac{1-\alpha^2}{\beta} & 1+\alpha \end{bmatrix} \right).$$

Now, Eq. (27) and Eq. (28) yield:

$$\begin{cases} \lambda_1^{(n+1)} = a\lambda_1^{(n)} \left(1 - \lambda_1^{(n)} \right); \\ \lambda_2^{(n+1)} = a\lambda_2^{(n)} \left(1 - \lambda_2^{(n)} \right); \end{cases} \quad n = 0, 1, 2, \dots \tag{29}$$

and

$$\begin{cases} x_{11}^{(n)} = \frac{1}{2} \left(\lambda_1^{(n)} + \lambda_2^{(n)} \right) + \frac{\alpha}{2} \left(\lambda_1^{(n)} - \lambda_2^{(n)} \right); \\ x_{12}^{(n)} = \frac{1}{2}\beta \left(\lambda_1^{(n)} - \lambda_2^{(n)} \right); \\ x_{21}^{(n)} = \frac{1}{2} \cdot \frac{1-\alpha^2}{\beta} \left(\lambda_1^{(n)} - \lambda_2^{(n)} \right); \\ x_{22}^{(n)} = \frac{1}{2} \left(\lambda_1^{(n)} + \lambda_2^{(n)} \right) - \frac{\alpha}{2} \left(\lambda_1^{(n)} - \lambda_2^{(n)} \right); \end{cases} \quad n = 0, 1, 2, \dots. \tag{30}$$

where $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$ are eigenvalues of $X^{(n)}$. It is clear that if $0 \le \lambda_1^{(0)}, \lambda_2^{(0)} \le 1$ and $0 \le a \le 4$ then $0 \le \lambda_1^{(n)}, \lambda_2^{(n)} \le 1$ for all $n = 1, 2, \dots$ what is a sufficient condition for $x_{kl}^{(n)}$; $k, l = 1, 2$ to be bounded for all $n = 0, 1, 2, \dots$. $\square$

Theorem 3.2. *Four iterated sequences* $\left\{ x_{kl}^{(n)} \right\}_{n=0}^{+\infty}$; $k, l = 1, 2$ *generated by the logistic map of matrices defined by Eq.* (2) *will stay bounded for all* $n = 0, 1, 2, \dots$ *if following statements hold:*

(i) $0 \le a \le 4$;

(ii) *the matrix of initial conditions* $X^{(0)}$ *is a nilpotent matrix;*

(iii) *the eigenvalue* $\lambda_0^{(0)}$ *of* $X^{(0)}$ *is bounded in the interval* $[0; 1]$;

(iv) *elements of the sequence* $\left\{ a^{n+1} \prod_{k=0}^{n} \left(1 - 2\lambda_0^{(k)} \right) \right\}_{n=0}^{+\infty}$ *are bounded in the interval* $[-M; M]$; $0 \le M < +\infty$; *where* $\lambda_0^{(k)}$ *is the recurrent eigenvalue of* $X^{(k)}$; $k = 0, 1, 2, \dots$.

Proof. Since $X^{(0)}$ is a nilpotent matrix:

$$X^{(0)} = \begin{bmatrix} x_{11}^{(0)} & x_{12}^{(0)} \\ x_{21}^{(0)} & x_{22}^{(0)} \end{bmatrix} = \lambda_0^{(0)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\frac{\hat{\alpha}^2}{\hat{\beta}} & -\hat{\alpha} \end{bmatrix}. \tag{31}$$

The introduction of Eq. (31) into Eq. (2) produces an iterative sequence of matrices; balancing appropriate components yields two scalar maps:

$$\begin{cases} \lambda_0^{(n+1)} = a\lambda_0^{(n)} \left(1 - \lambda_0^{(n)} \right); \\ \mu_0^{(n+1)} = a\mu_0^{(n)} \left(1 - 2\lambda_0^{(n)} \right); \end{cases} \quad n = 0, 1, 2, \dots \tag{32}$$

where $\mu_0^{(0)} = 1$ and $\left\{\mu_0^{(n)}\right\}_{n=1}^{+\infty}$ are coefficients at the nilpotent $\frac{1}{2}\begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\frac{\hat{\alpha}^2}{\hat{\beta}} & -\hat{\alpha} \end{bmatrix}$

in the decomposition of the nilpotent matrices $\left\{X^{(n)}\right\}_{n=1}^{+\infty}$. Now, four iterative sequences take the following form:

$$\begin{cases} x_{11}^{(n)} = \lambda_0^{(n)} + \frac{\hat{\alpha}}{2}\mu_0^{(n)}; \\ x_{12}^{(n)} = \frac{\hat{\beta}}{2}\mu_0^{(n)}; \\ x_{21}^{(n)} = -\frac{\hat{\alpha}^2}{2\hat{\beta}}\mu_0^{(n)}; \qquad n = 0, 1, 2, \ldots \\ x_{22}^{(n)} = \lambda_0^{(n)} - \frac{\hat{\alpha}}{2}\mu_0^{(n)}; \end{cases} \qquad (33)$$

Now, it is sufficient that $0 \le \lambda_0^{(0)} \le 1$; $0 \le a \le 4$ and $\left|\mu_0^{(n)}\right| < M < +\infty$ for $x_{kl}^{(n)}$; $k, l = 1, 2$ to be bounded for all $n = 1, 2, \ldots$. On the other hand, the boundedness of coefficients $\left\{\mu_0^{(n)}\right\}_{n=1}^{+\infty}$ is determined by Eq. (32):

$$\mu_0^{(n+1)} = a^{n+1}\prod_{k=0}^{n}\left(1 - 2\lambda_0^{(k)}\right); \quad n = 0, 1, 2, \ldots. \qquad (34)$$

$\square$

Corollary 10. *This corollary describes the logistic evolution of square matrices of order 2 from initial conditions.*

  (i) *If the matrix of initial conditions $X^{(0)}$ is a scalar matrix then iterated matrices $X^{(n)}$ will stay scalar matrices for all $n = 1, 2, \ldots$.*
 (ii) *If the matrix of initial conditions $X^{(0)}$ is an idempotent matrix then iterated matrices $X^{(n)}$ stay idempotent matrices for all $n = 1, 2, \ldots$ or may become scalar matrices from $n = m, m+1, \ldots; m \ge 1$.*
(iii) *If the matrix of initial conditions $X^{(0)}$ is a nilpotent matrix then iterated matrices $X^{(n)}$ stay nilpotent matrices for all $n = 1, 2, \ldots$ or may become scalar matrices from $n = m, m+1, \ldots; m \ge 1$.*

Proof. (i) If the matrix of initial conditions is a scalar matrix $X^{(0)} = \lambda_0^{(0)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

then iterated matrices stay scalar matrices $X^{(n)} = \lambda_0^{(n)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for all $n = 1, 2, \ldots$

because $\lambda_0^{(n+1)} = a\lambda_0^{(n)}(1 - \lambda_0^{(n)})$.

(ii) If the matrix of initial conditions $X^{(0)}$ is an idempotent matrix then iterated matrices $X^{(n)}$ stay idempotent matrices:

$$X^{(n)} = \lambda_1^{(n)}\frac{1}{2}\begin{bmatrix} 1+\alpha & \beta \\ \frac{1-\alpha^2}{\beta} & 1-\alpha \end{bmatrix} + \lambda_2^{(n)}\frac{1}{2}\begin{bmatrix} 1-\alpha & -\beta \\ -\frac{1-\alpha^2}{\beta} & 1+\alpha \end{bmatrix} \qquad (35)$$

until $\lambda_1^{(n)} \ne \lambda_2^{(n)}$ for $n = 1, 2, \ldots$ (Theorem 3.1). But if eigenvalues of the iterated matrix $X^{(m)}$ become equal at $m > 0$: $\lambda_1^{(m)} = \lambda_2^{(m)} = \lambda_0^{(m)}$ then iterated matrices $X^{(k)}$ become scalar matrices: $X^{(k)} = \lambda_0^{(k)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $k = m, m+1, m+2, \ldots$.

(iii) If the matrix of initial conditions $X^{(0)}$ is a nilpotent matrix then iterated matrices $X^{(n)}$ stay nilpotent matrices (Theorem 3.2):

$$X^{(n)} = \lambda_0^{(n)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mu_0^{(n)} \frac{1}{2} \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ -\frac{\hat{\alpha}^2}{\hat{\beta}} & -\hat{\alpha} \end{bmatrix} \qquad (36)$$

until $\mu_0^{(n)} \neq 0$ for $n = 1, 2, \ldots$. But if $\mu_0^{(m)}$ becomes equal to 0 at $m > 0$ then iterated matrices $X^{(k)}$ become scalar matrices: $X^{(k)} = \lambda_0^{(k)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $k = m, m+1, m+2, \ldots$

$\square$

3.2. **Computational experiments.** First of all it can be noted that the qualitative behavior of iterated matrices of order 2 is governed by Eq. (29) or Eq. (32) (depending from the type of the matrix of initial conditions $X^{(0)}$). Secondly, it is important to stress that the evolution of the map differs substantially if $X^{(0)}$ is an idempotent or a nilpotent matrix. If $X^{(0)}$ has two distinct eigenvalues in $[0; 1]$, it is a sufficient condition that the elements of iterated matrices would be bounded for $0 \leq a \leq 4$. But if $X^{(0)}$ has one recurrent eigenvalue in $[0; 1]$, one can be sure that the elements of iterated matrices would be bounded only for $0 \leq a \leq 1$; a separate investigation must be done for higher values of the parameter a.

3.2.1. *Asymptotic versus nonasymptotic convergence; $1 < a < 3$.* Let the matrix of initial conditions is an idempotent matrix and the parameter of the logistic map a is bounded in the interval $1 < a < 3$ (a scalar logistic map converges to a stable fixed point $1 - a^{-1}$ then). But then, according to the system of equations (29), both eigenvalues $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$ will converge to $1 - a^{-1}$ at increasing n (if, of course, $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$ are bounded in the interval $[0; 1]$). In other words, the idempotent matrix of initial conditions will eventually be transformed into a scalar matrix at sufficiently high n. But such a transformation requires additional explanations which are given below.

First of all it can be noted that the convergence of a scalar logistic map to a stable fixed point $1 - a^{-1}$ can be asymptotic or nonasymptotic. Let us assume that a current state of the scalar logistic map (Eq. (1)) is $x^{(n)}$. Then a backward iteration from $x^{(n)}$ can be described by the following equality:

$$\left(x^{(n-1)} \right)_{1,2} = \frac{1}{2} \left(1 \pm \sqrt{1 - \frac{4}{a} x^{(n)}} \right); \qquad (37)$$

where the necessary condition for the backward iteration is

$$a - 4x^{(n)} \geq 0. \qquad (38)$$

Such a backward iterative process generates a backward tree of points (some branches of the tree are cut as the requirement (38) may not always hold) [21]. Therefore there exist such points which would yield the exact value of the stable fixed point $1 - a^{-1}$ in a finite number of forward iterations (nonasymptotic convergence) [22]. All other initial conditions (in the interval $[0; 1]$) converge to the fixed point asymptotically.

Fig. 2 is used to illustrate asymptotic and nonasymptotic convergence of eigenvalues to a fixed point at $a = 2.5$. The idempotent matrix of initial conditions $X^{(0)} = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$ is gradually transformed into a scalar matrix: $\lim_{n \to \infty} X^{(n)} =$

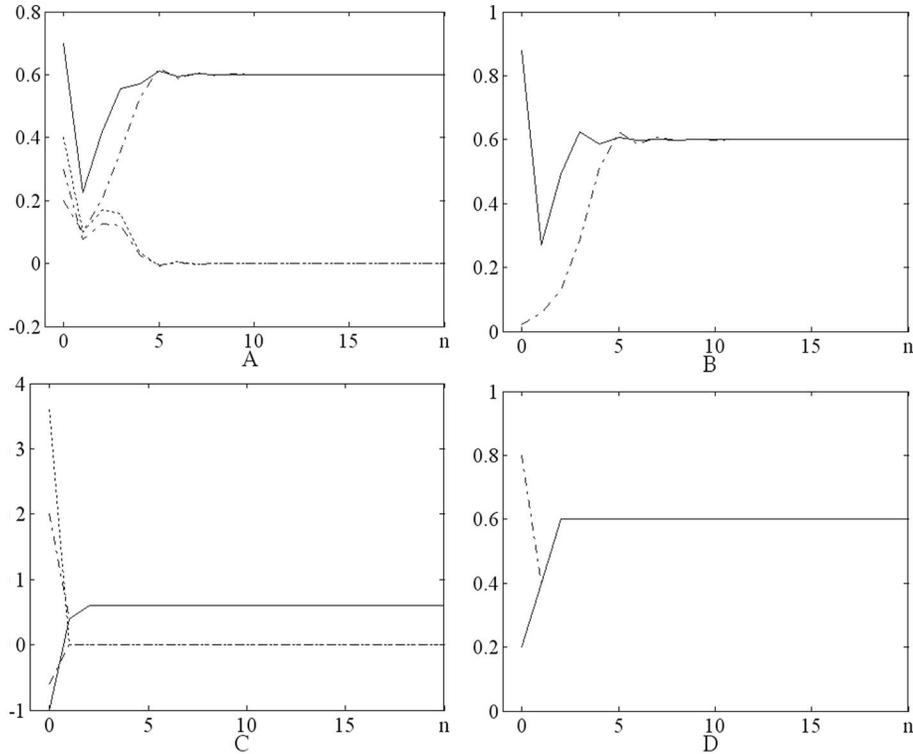FIGURE 2. Asymptotic versus nonasymptotic convergence to a
period-1 attractor: $X^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{smallmatrix}\right]$ results into asymptotic con-
vergence (A showing the evolution of $x_{11}^{(n)}$, $x_{12}^{(n)}$, $x_{21}^{(n)}$, $x_{22}^{(n)}$ and B
showing the evolution of eigenvalues $\lambda_1^{(n)}$, $\lambda_2^{(n)}$); $X^{(0)} = \left[\begin{smallmatrix} 2 & -0.6 \\ 3.6 & -1 \end{smallmatrix}\right]$
results into nonasymptotic convergence (C showing the evolution
of elements of the matrix and D showing the evolution of its eigen-
values); $a = 2.5$ in both experiments.

$\begin{bmatrix} 0.6 & 0 \\ 0 & 0.6 \end{bmatrix}$ (Fig. 2A), while its eigenvalues $\lambda_1^{(0)} = 0.023$ and $\lambda_2^{(0)} = 0.877$ con-
verge asymptotically to the fixed point $1 - a^{-1} = 0.6$ (Fig. 2B). Alternatively, the
idempotent matrix of initial conditions $X^{(0)} = \begin{bmatrix} 2 & -0.6 \\ 3.6 & -1 \end{bmatrix}$ is transformed into
a scalar matrix in two steps: $X^{(2)} = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.6 \end{bmatrix}$ (Fig. 2C) while its eigenvalues
$\lambda_1^{(0)} = 0.2$ and $\lambda_2^{(0)} = 0.8$ converge nonasymptotically to 0.6 (Fig. 2D): $\lambda_1^{(1)} = 0.4$;
$\lambda_2^{(1)} = 0.4$; $\lambda_1^{(2)} = 0.6$; $\lambda_2^{(2)} = 0.6$.

It can be noted that only two backward iterations were used to construct eigen-
values of $X^{(0)}$ in this computational example. Of course, more complex examples
of nonasymptotic convergence could be used to illustrate the transition from an
idempotent matrix to a scalar matrix. In general if eigenvalues of the matrix $X^{(n)}$
are $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$ a backward iteration reads:

$$\begin{cases} \left(\lambda_1^{(n-1)}\right)_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{a}\lambda_1^{(n)}}\right); \\ \left(\lambda_2^{(n-1)}\right)_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{a}\lambda_2^{(n)}}\right). \end{cases} \tag{39}$$

It can be noted that a backward iteration is possible only when $a - 4\lambda_1^{(n)} \geq 0$ and $a - 4\lambda_2^{(n)} \geq 0$.

If $X^{(n)}$ is a nilpotent matrix, a backward iteration reads:

$$\begin{cases} \left(\lambda_0^{(n-1)}\right)_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{a}\lambda_0^{(n)}}\right); \\ \left(\mu_0^{(n-1)}\right)_{1,2} = \frac{1}{a\left(1 - 2\left(\lambda_0^{(n-1)}\right)_{1,2}\right)}\left(\mu_0^{(n)}\right)_{1,2}; \end{cases} \tag{40}$$

and the necessary conditions for a backward iteration are then:

$$\begin{cases} a - 4\lambda_0^{(n)} \geq 0; \\ 0 \leq \left(\lambda_0^{(n-1)}\right)_{1,2} < 1. \end{cases} \tag{41}$$

3.2.2. *Periodic attractors at $a = 3.55$; $X^{(0)}$ is an idempotent matrix.* A period-4 stable attractor exists in a scalar logistic map at $a = 3.55$ (the convergence to this attractor again can be asymptotic or nonasymptotic). Then the following question arises: will any idempotent matrix of initial conditions evolve into a scalar matrix when eigenvalues will be gradually (or in a finite number of steps) attracted to the period-4 attractor (eigenvalues of $X^{(0)}$ are bounded in $[0;1]$ of course)?

The answer is negative. Eigenvalues of $X^{(0)}$ will be attracted to the period-4 attractor in any case, but a phase difference between iterated eigenvalues can be not necessarily equal to zero. This phase difference is constant (and can be equal to 0, 1, 2 or 3 iterates) when both eigenvalues are in the period-4 regime. For example, an idempotent matrix $X^{(0)} = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$ is gradually transformed into a sequence of scalar matrices (4 different scalar matrices in a period) (Fig. 3A) while its eigenvalues asymptotically converge to the period-4 attractor without a phase difference (Fig. 3B).

But the idempotent matrix $X^{(0)} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \cdot \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} -1.1 & 0.6 \\ -2.8 & 1.5 \end{bmatrix}$ yields an infinite sequence of idempotent matrices because its eigenvalues converge to the period-4 attractor with a constant phase difference not equal to 0 (Fig. 3C and Fig. 3D).

3.2.3. *The evolution of the logistic map of matrices when $X^{(0)}$ is a nilpotent matrix.* A nilpotent matrix of initial conditions $X^{(0)}$ defined by Eq. (31) will be considered in this section. Values of parameters $\lambda_0^{(0)} = 0.3$; $\hat{\alpha} = 2$ and $\hat{\beta} = 8$ yield $X^{(0)} = \begin{bmatrix} 1.3 & 4 \\ -0.25 & -0.7 \end{bmatrix}$. Fig. 4A and Fig. 4B show strong fluctuations of four scalar time series (Eq. (2)) and appropriate eigenvalues in the interval $0 \leq n \leq 50$, but the processes calm down at higher n. Particularly, Fig. 4C shows that iterated matrices become scalar matrices. Fig. 4D shows that eigenvalues $\lambda_0^{(n)}$ oscillate in the interval
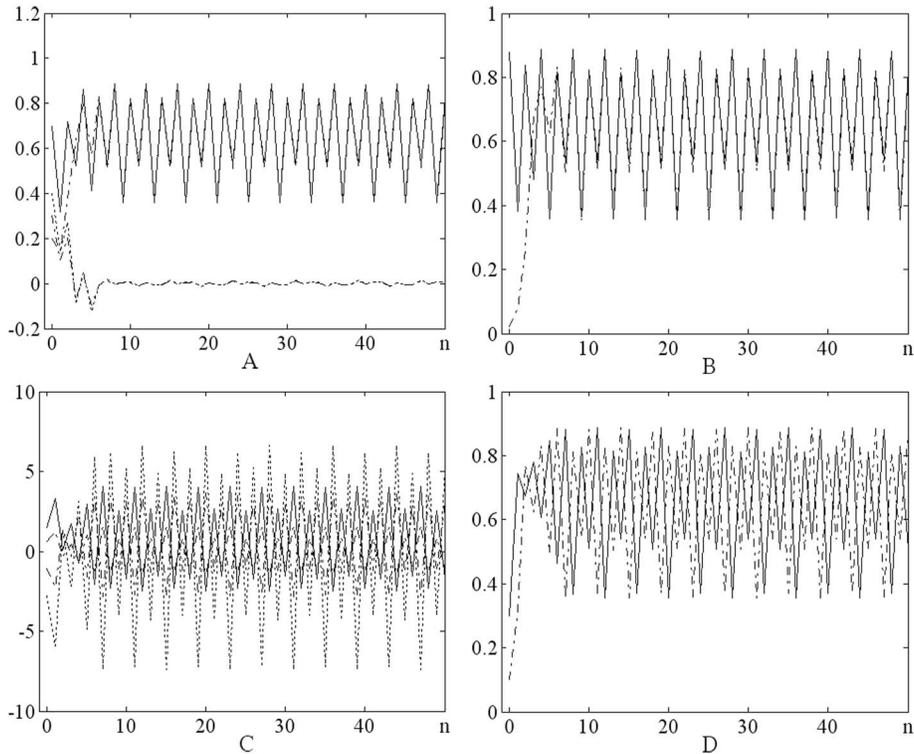
FIGURE 3. An idempotent matrix of initial conditions can yield a sequence of scalar matrices or a sequence of idempotent matrices: $X^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{smallmatrix}\right]$ converges to a sequence of scalar matrices (A showing the evolution of elements of the matrix and B showing the evolution of its eigenvalues) – the phase difference between eigenvalues in the period-4 regime is equal to 0; $X^{(0)} = \left[\begin{smallmatrix} -1.1 & 0.6 \\ -2.8 & 1.5 \end{smallmatrix}\right]$ yields an infinite sequence of idempotent matrices (C showing the evolution of elements of the matrix and D showing the evolution of its eigenvalues) because eigenvalues converge to the period-4 regime with a phase difference; $a = 3.55$ in both experiments.

between 0 and 1 what is a necessary (but not a sufficient) condition of convergence of the product in Eq. (32). It is interesting to note that parameters $\mu_0^{(n)}$ tend to zero thus ensuring the boundedness of $\left\{ x_{kl}^{(n)} \right\}_{n=0}^{+\infty}$; $k, l = 1, 2$. A different value of the parameter a ($a = 3.6$) yields a violent divergence of iterative processes (Fig. 5).

3.2.4. *The sensitivity to initial conditions at $a = 3.7$.* It is well known that a scalar logistic map evolves to chaos after a cascade of period doubling bifurcations. At $a = 3.7$ the dynamics of a scalar logistic map is already chaotic. The sensitivity to initial conditions is one of the characteristic features of the deterministic chaos [2]. We will illustrate this feature using the logistic map of matrices.
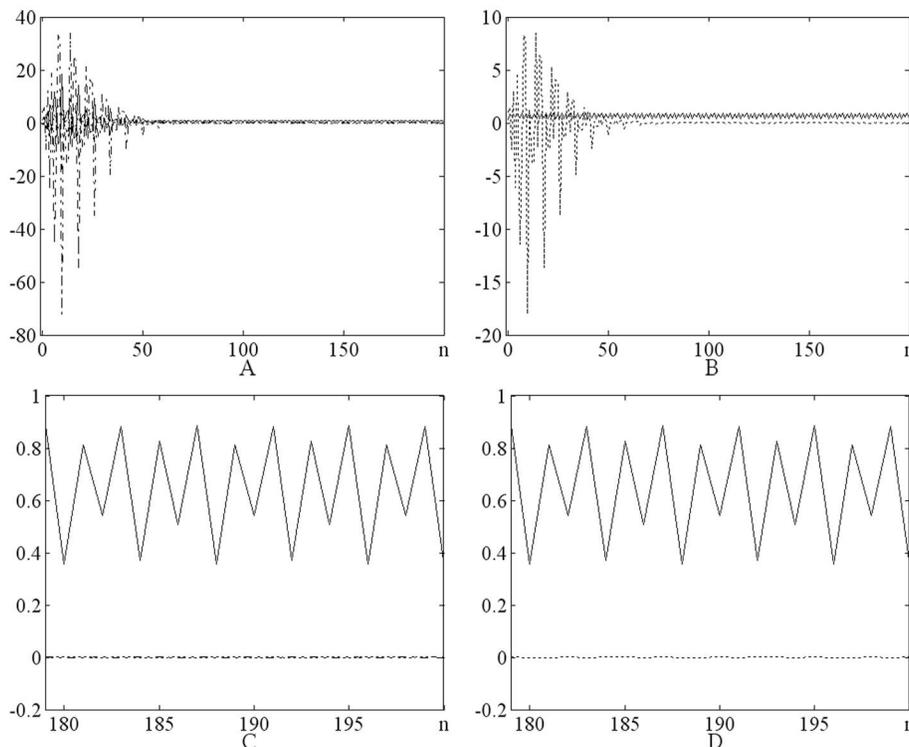
FIGURE 4. The evolution of the logistic map of matrices from
$X^{(0)} = \begin{bmatrix} 1.3 & 4 \\ -0.25 & -0.7 \end{bmatrix}$ at $a = 3.55$ (A and C showing the evolution of
$x_{11}^{(n)}$, $x_{12}^{(n)}$, $x_{21}^{(n)}$, $x_{22}^{(n)}$; B and D showing the evolution of the eigen-
value (a solid line) and the parameter $\mu_0^{(n)}$) defined by Eq. (30) (a
dashed line). Evolutions in C and D are displayed in the interval
$180 \leq n \leq 200$ where n is the iteration number.

The matrix of initial conditions $X^{(0)} = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$ and its eigenvalues yield
chaotic sequences at $a = 3.7$ (Fig. 6A and Fig. 6B). We construct a perturbed
matrix of initial conditions $X^{(0)} = \begin{bmatrix} 0.2 + \varepsilon & 0.3 + \varepsilon \\ 0.4 + \varepsilon & 0.7 + \varepsilon \end{bmatrix}$; $\varepsilon = 10^{-6}$ and follow the
iterative processes (Fig. 6C and Fig. 6D). Differences between values of iterated
elements and iterated eigenvalues of these matrices are shown in Fig. 6E and Fig. 6F.

4. **Concluding Remarks.** The standard logistic map is extended by replacing the
scalar iterative variable by a square matrix of variables. Main dynamical features
of this iterative map are discussed in detail and illustrated by numerical examples.

One of many applications of the standard logistic map - the encryption of optical
images - is discussed in the Introduction. The simplest approach comprises 2 iter-
ative logistic maps used to generate coordinates of encrypted points (the first one
generates x-coordinates; the second one - y-coordinates of a sequence of points); 4
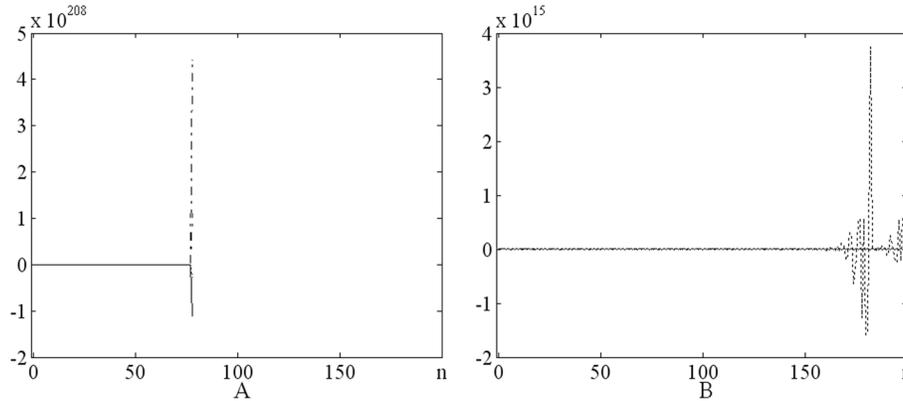keys are necessary to initiate the process (2 initial conditions and 2 values of the

FIGURE 5. The evolution of the logistic map of matrices from $X^{(0)} = \begin{bmatrix} 1.3 & 4 \\ -0.25 & -0.7 \end{bmatrix}$ at $a = 3.6$ (A showing the evolution of $x_{11}^{(n)}$, $x_{12}^{(n)}$, $x_{21}^{(n)}$, $x_{22}^{(n)}$; B showing the evolution of the eigenvalue (a solid line) and the parameter $\mu_0^{(n)}$) defined by Eq. (30) (a dashed line).

parameter a). A single 2-dimensional logistic map of matrices could be used instead. 5 keys would be necessary now to initiate the process (4 initial conditions and 1 value of the parameter a). But the safety of such an approach and its robustness to a brute force attack would be considerably higher compared to the solution based on 2 separate standard logistic maps - in many cases initial conditions would lead to a violent divergence or quenching of the iterative process. Of course, detailed analysis of such an encryption technique is an object of future research.

It appears that the dynamics of the logistic map of matrices of order 2 can be interpreted exploiting the concept of independent one dimensional logistic maps of the eigenvalues of the matrix X_0. Two independent scalar logistic maps govern the dynamics of the logistic map of matrices if the matrix of initial conditions is an idempotent matrix. Alternatively, one scalar logistic map and another special (non-logistic) iterative relationship govern the evolution of the system if the matrix of initial conditions is a nilpotent matrix. These results are not trivial at all and explain complex behavior of this relatively simple dynamical system.

Nevertheless, a number of open questions remain. The first question is about the size of the matrix – what are dynamical properties of the logistic map of square matrices of order n. The second question is regarding the convergence of the logistic map of matrices. Clocking convergence is a well explored topic in nonlinear dynamics of the standard logistic map; the Lyapunov exponent is used to characterize its stability. The computation of the spectrum of Lyapunov exponents and the measurement of the speed of convergence towards a stable attractor for the logistic map of matrices are rather subtle problems. It is not enough to perturb eigenvalues (consider the nilpotent matrix of initial conditions). The third (and probably the most important) question is about the potential of applicability of the logistic map of matrices for similar problems the standard logistic map has been used to – modeling, encryption and prediction of different systems and processes. All these questions pose a definite interest and are an object of the future research.
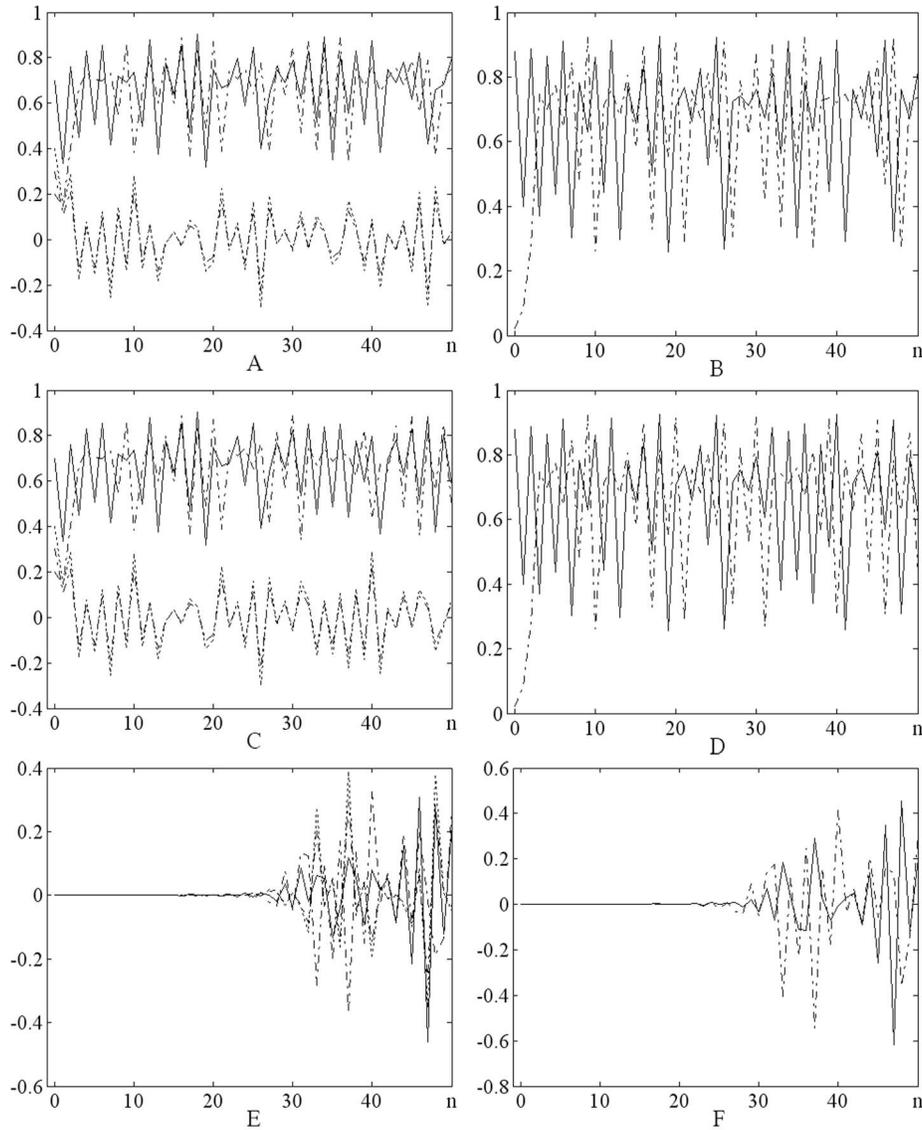
FIGURE 6. The illustration of the sensitivity to initial conditions at $a = 3.7$; $X^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{smallmatrix}\right]$ yields chaotic sequences (A – the evolution of elements; B – the evolution of eigenvalues); $X^{(0)} = \left[\begin{smallmatrix} 0.2+\varepsilon & 0.3+\varepsilon \\ 0.4+\varepsilon & 0.7+\varepsilon \end{smallmatrix}\right]$; $\varepsilon = 10^{-6}$ also yields chaotic sequences (C – the evolution of elements; D – the evolution of eigenvalues); E shows differences between appropriate elements of matrices; F shows differences between appropriate eigenvalues.

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